Question

Evaluate $\int x \sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx$

Solution

$$\int x \sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx = \frac{1}{2}\int \sqrt{\frac{a^2-x^2}{a^2+x^2}}\cdot 2x\, dx = \left\{ \begin{array}{l} \sqrt{\dfrac{a^2-x^2}{a^2+x^2}} = t,\ a^2-x^2 = a^2t^2 + x^2t^2, \\[2mm] x^2 = \dfrac{a^2(1-t^2)}{1+t^2},\ \ 2x\,dx = a^2 \cdot \dfrac{-4t\,dt}{(1+t^2)^2} \end{array} \right\} =$$

$$= \frac{1}{2}\int t \cdot a^2 \cdot \frac{-4t\,dt}{(1+t^2)^2} = a^2 \int t \cdot \frac{-2t\,dt}{(1+t^2)^2} = a^2 \int t \cdot d\left(\frac{1}{1+t^2}\right) = a^2 t \cdot \frac{1}{1+t^2} -$$

$$-a^2 \int \frac{1}{1+t^2}\, dt = \frac{a^2 t}{1+t^2} - a^2 \cdot \arctan(t) + c = \left\{ \sqrt{\frac{a^2-x^2}{a^2+x^2}} = t \right\} = a^2 \cdot \frac{\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}}{1+\dfrac{a^2-x^2}{a^2+x^2}} -$$

$$-a^2 \cdot \arctan\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right) + c = \frac{\sqrt{a^4-x^4}}{2} - a^2 \cdot \arctan\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right) + c.$$

Answer: $\int x \sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx = \frac{\sqrt{a^4-x^4}}{2} - a^2 \cdot \arctan\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right) + c.$