Answer on Question #84831 – Math – Calculus

Question

Prove for every $x > 0$ $\quad \frac{x}{1+x^2} < \tan^{-1} x < x$

Solution

Let us consider an inequality 4.3.80 on the page 75 from (*)

$$\sin t < t < \tan t \ \ \forall t \in \left(0; \frac{\pi}{2}\right)$$

So

$$\sin t < t \ \ \forall t \in \left(0; \frac{\pi}{2}\right) \qquad (1)$$

$$t < \tan t \ \ \forall t \in \left(0; \frac{\pi}{2}\right) \qquad (2)$$

Let us change the variable $t = \tan^{-1} x\,, \quad x > 0$ in (1)

$$\tan^{-1} x < \tan(\tan^{-1} x) \quad \forall x > 0$$

So that

$$\tan^{-1} x < x \ \ \forall x > 0 \quad (1')$$

Let us change the variable $t = 2\tan^{-1} x\,, \quad x > 0$ in (2)

$$\sin(2\tan^{-1} x) < 2\tan^{-1} x\,, \quad x > 0$$

Using $\left[\sin 2\alpha = \frac{2\tan\alpha}{1+(\tan\alpha)^2}\right]$ in previous formula we obtain

$$\sin(2\tan^{-1} x) = \frac{2\tan(\tan^{-1} x)}{1+\left(\tan(\tan^{-1} x)\right)^2} = \frac{2x}{1+x^2} < 2\tan^{-1} x\,, \quad x > 0$$

So that

$$\frac{x}{1+x^2} < \tan^{-1} x\,, \quad x > 0 \quad (2')$$

$$(1'), (2') \sim \quad \frac{x}{1+x^2} < \tan^{-1} x < x \quad \forall x > 0 \ ■$$

(*) Milton Abramowitz and Irene Stegun (1972), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* United States Department of Commerce, National Bureau of Standards; ISBN 0-486-61272-4.