Answer on Question #84830 – Math – Calculus

Question

Show that limit n ->∞ [(1/n) + 1/(n+1) +1/(n+2) + ...+ (1/3n)]= log3

Solution

Partial sum of the first m members of the harmonic series:

$$H_m = \sum_{k=1}^{m} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} = \ln m + \gamma + \varepsilon_m,$$

$$where\ \gamma = 0.5772\ldots - the\ Euler\text{–}Mascheroni\ constant$$

$$if\ m \to \infty\ then\ \varepsilon_m \to 0\ and\ H_m \approx \ln m + \gamma \qquad (1)$$

$$for\ \ m = 3n \to H_{3n} = \sum_{k=1}^{3n} \frac{1}{k}$$

$$for\ \ m = n \to H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n} = \sum_{k=1}^{3n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n} = H_{3n} - H_n + \frac{1}{n}$$

So

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right) = \lim_{n\to\infty} \left(H_{3n} - H_n + \frac{1}{n}\right) = \lim_{n\to\infty} (H_{3n}) - \lim_{n\to\infty} (H_n) + \lim_{n\to\infty} \left(\frac{1}{n}\right)$$
$$= \left\{(1) \to \lim_{m\to\infty} H_m = \ln m + \gamma\right\} = \ln 3n + \gamma - \ln n - \gamma + 0 = \ln 3n - \ln n = \ln \frac{3n}{n}$$
$$= \ln 3$$

i.e.

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right) = \ln 3$$