Answer on Question #84514 – Math – Complex Analysis

Question

If $f_1(z)$ and $f_2(z)$ are two entire functions of order r_1 and r_2 respectively, then show that order of
(i) $f_1(z) \pm f_2(z) \leq \max\{r_1,\ r_2\}$
(ii) $f_1(z) * f_2(z) \leq \max\{r_1,\ r_2\}$.

Solution

By definition, an entire function f is of finite order $r > 0$ if for (every positive but not negative) $0 \not< e > 0$ there is $w > 0$ such that $|f(z)| < \exp(|z|^{r+e})$ for $|z| > w$.
We have that
(i)
$|f_1(z) \pm f_2(z)| \leq |f_1(z)| + |f_2(z)| < 2\exp\left(|z|^{\max\{r_1+e, r_2+e\}}\right) = \exp\left(\ln 2 + |z|^{\max\{r_1,r_2\}+e}\right) < \exp\left(2 * |z|^{\max\{r_1,r_2\}+e}\right) < \exp\left(|z|^{\max\{r_1,r_2\}+2e}\right)$ for $|z| > \max\{w_1,\ w_2,\ w(\ln 2\,, r_1, r_2, e)\}$
(ii)
$|f_1(z) * f_2(z)| = |f_1(z)| * |f_2(z)| = \exp(|z|^{r_1+e} + |z|^{r_2+e}) < \exp\left(2 * |z|^{\max\{r_1,r_2\}+e}\right) < \exp\left(|z|^{\max\{r_1,r_2\}+2e}\right)$ for all $e > 0$ and $|z| > \max\{w_1,\ w_2,\ w(e)\}$

The proof depends on the definition. We may assume equality in (i) if $r_1 \neq r_2$.

References:
Holland, Anthony S B (1973). *An introduction to the theory of entire functions*. pp 59-61.