## Answer on Question #84507 – Math – Differential Equations

## Question

Using Rodrigue's formula, obtain the expression for the Hermite polynomial $H_4(x)$.

## Solution

Hermite polynomials, named after the French mathematician Charles Hermite, are orthogonal polynomials, in a sense to be described below, of the form

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

for $n = 0, 1, 2, 3, \dots$

This is the Rodrigues formula for the Hermite polynomial.

for $n = 0$: $H_0(x) = (-1)^0 e^{x^2}\left(e^{-x^2}\right) = 1$

for $n = 1$: $H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) = -1(-2x)e^{x^2}\left(e^{-x^2}\right) = 2x$

for $n = 2$: $H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2}\right) =$

$= e^{x^2}\left(-2e^{-x^2} - 2x(-2x)e^{-x^2}\right) = 4x^2 - 2$

for $n = 3$: $H_3(x) = (-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right) = -e^{x^2} \frac{d^2}{dx^2}\left(-2xe^{-x^2}\right) =$

$= -e^{x^2} \frac{d}{dx}\left(-2e^{-x^2} - 2x(-2x)e^{-x^2}\right) =$

$= -e^{x^2}\left(-2(-2x)e^{-x^2} + 8xe^{-x^2} + 4x^2(-2x)e^{-x^2}\right) =$

$= 8x^3 - 12x$

for $n = 4$: $H_4(x) = (-1)^4 e^{x^2} \frac{d^4}{dx^4}\left(e^{-x^2}\right) = e^{x^2} \frac{d^3}{dx^3}\left(-2xe^{-x^2}\right) =$

$= e^{x^2} \frac{d^2}{dx^2}\left(-2e^{-x^2} - 2x(-2x)e^{-x^2}\right) =$

$= e^{x^2} \frac{d}{dx}\left(-2(-2x)e^{-x^2} + 8xe^{-x^2} + 4x^2(-2x)e^{-x^2}\right) =$

$= e^{x^2}\left(12e^{-x^2} + 12x(-2x)e^{-x^2} - 24x^2e^{-x^2} - 8x^3(-2x)e^{-x^2}\right) =$

$= 16x^4 - 48x^2 + 12$

Recurrence relations

1. $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \qquad n = 1, 2, \dots$
2. $H_n'(x) = 2nH_{n-1}(x) \qquad n = 1, 2, \dots$
   $H_{n+1}'(x) = 2(n+1)H_n(x) \qquad n = 1, 2, \dots$
3. $H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0 \qquad n = 0, 1, 2, \dots$