ANSWER on Question #84506 – Math – Differential Equations

QUESTION

Show that

1)

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos(x)$$

2)

$$\frac{d}{dx}\left(x \cdot J_1(x)\right) = x \cdot J_0(x)$$

where $J_x(x)$ is Bessel function.

(More information: https://en.wikipedia.org/wiki/Bessel_function)

SOLUTION

2) To prove this formula, we will prove the general formulas:

$$\begin{cases} J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} \cdot J_n(x) \\ J_{n-1}(x) - J_{n+1}(x) = 2 \cdot J_n'(x) \end{cases} \rightarrow \frac{d}{dx}\left(x^n \cdot J_n(x)\right) = x^n \cdot J_{n-1}(x)$$

For this we use the generating function:

$$e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n \cdot J_n(x)$$

(More information: https://en.wikipedia.org/wiki/Bessel_function)

2 FORMULA:

$$\frac{d}{dx}\Bigg| e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n \cdot J_n(x) \rightarrow$$

$$e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} \cdot \frac{1}{2}\left(t-\frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} t^n \cdot J_n'(x) \rightarrow \left(\sum_{n=-\infty}^{\infty} t^n \cdot J_n(x)\right) \cdot \frac{1}{2}\left(t-\frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} t^n \cdot J_n'(x) \Bigg| \cdot (2)$$

$$\left(\sum_{n=-\infty}^{\infty} t^n \cdot J_n(x)\right) \cdot \left(t-\frac{1}{t}\right) = 2 \cdot \sum_{n=-\infty}^{\infty} t^n \cdot J_n'(x) \rightarrow$$

$$\sum_{n=-\infty}^{\infty} t^{n+1} \cdot J_n(x) - \sum_{n=-\infty}^{\infty} t^{n-1} \cdot J_n(x) = 2 \cdot \sum_{n=-\infty}^{\infty} t^n \cdot J_n'(x)$$

Since this is an equality of power series, the coefficients with the same t^n must be the same. To make it easier to compare the coefficients, we will do the remapping

$$n+1 = k \rightarrow \sum_{n=-\infty}^{\infty} t^{n+1} \cdot J_n(x) = \sum_{k=-\infty}^{\infty} t^k \cdot J_{k-1}(x) \equiv$$

$$n-1 = m \rightarrow \sum_{n=-\infty}^{\infty} t^{n-1} \cdot J_n(x) = \sum_{m=-\infty}^{\infty} t^m \cdot J_{m+1}(x) \equiv \sum_{n=-\infty}^{\infty} t^n \cdot J_{n+1}(x)$$

Then,

$$\sum_{n=-\infty}^{\infty} t^n \cdot J_{n-1}(x) - \sum_{n=-\infty}^{\infty} t^n \cdot J_{n+1}(x) = 2 \cdot \sum_{n=-\infty}^{\infty} t^n \cdot J_n'(x) \rightarrow$$

$$\boxed{J_{n-1}(x) - J_{n+1}(x) = 2 \cdot J_n'(x)}$$

1 FORMULA:

$$\frac{d}{dt}\Bigg| e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n \cdot J_n(x) \to$$

$$e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} \cdot \frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} nt^{n-1} \cdot J_n(x) \to \left(\sum_{n=-\infty}^{\infty} t^n \cdot J_n(x)\right) \cdot \frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} nt^{n-1} \cdot J_n(x) \to$$

$$\frac{x}{2} \cdot \left(\sum_{n=-\infty}^{\infty} t^n \cdot J_n(x) + \sum_{n=-\infty}^{\infty} t^{n-2} \cdot J_n(x)\right) = \sum_{n=-\infty}^{\infty} nt^{n-1} \cdot J_n(x) \Bigg| \cdot \left(\frac{2}{x}\right)$$

$$\sum_{n=-\infty}^{\infty} t^n \cdot J_n(x) + \sum_{n=-\infty}^{\infty} t^{n-2} \cdot J_n(x) = \sum_{n=-\infty}^{\infty} \left(\frac{2n}{x}\right) t^{n-1} \cdot J_n(x)$$

Since this is an equality of power series, the coefficients with the same t^n must be the same. To make it easier to compare the coefficients, we will do the remapping:

$$n-1=m \to \sum_{n=-\infty}^{\infty} t^n \cdot J_n(x) = \sum_{m=-\infty}^{\infty} t^{m+1} \cdot J_{m+1}(x) \equiv \sum_{n=-\infty}^{\infty} t^{n+1} \cdot J_{n+1}(x)$$

$$n-1=m \to \sum_{n=-\infty}^{\infty} t^{n-2} \cdot J_n(x) = \sum_{m=-\infty}^{\infty} t^{m-1} \cdot J_{m-1}(x) \equiv \sum_{n=-\infty}^{\infty} t^{n-1} \cdot J_{n-1}(x)$$

Then,

$$\sum_{n=-\infty}^{\infty} t^{n+1} \cdot J_{n+1}(x) + \sum_{n=-\infty}^{\infty} t^{n-1} \cdot J_{n-1}(x) = \sum_{n=-\infty}^{\infty} \left(\frac{2n}{x}\right) t^{n-1} \cdot J_n(x) \to$$

$$\boxed{J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} \cdot J_n(x)}$$

Conclusion,

$$\boxed{\begin{cases} J_{n-1}(x) - J_{n+1}(x) = 2 \cdot J_n'(x) \\ J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} \cdot J_n(x) \end{cases}}$$

We add and subtract the equation indicating system

$$\begin{cases} J_{n-1}(x) - J_{n+1}(x) = 2 \cdot J_n'(x) \\ J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} \cdot J_n(x) \end{cases} \rightarrow \begin{cases} 2J_{n-1}(x) = \dfrac{2n}{x} \cdot J_n(x) + 2 \cdot J_n'(x) \Big| \cdot \left(\dfrac{x}{2}\right) \\ 2J_{n+1}(x) = \dfrac{2n}{x} \cdot J_n(x) - 2 \cdot J_n'(x) \Big| \cdot \left(\dfrac{x}{2}\right) \end{cases} \rightarrow$$

$$\begin{cases} x \cdot J_{n-1}(x) = n \cdot J_n(x) + x \cdot J_n'(x) \\ x \cdot J_{n+1}(x) = n \cdot J_n(x) - x \cdot J_n'(x) \end{cases}$$

The first equation of the resulting system is important for us.

$$x \cdot J_{n-1}(x) = n \cdot J_n(x) + x \cdot J_n'(x) | \cdot (x^{n-1}) \rightarrow$$

$$x^n \cdot J_{n-1}(x) = n \cdot x^{n-1} \cdot J_n(x) + x^n \cdot J_n'(x) \rightarrow \begin{bmatrix} n \cdot x^{n-1} \equiv \dfrac{d}{dx}(x^n) \\ J_n'(x) \equiv \dfrac{d}{dx}\big(J_n(x)\big) \end{bmatrix} \rightarrow$$

$$x^n \cdot J_{n-1}(x) = \frac{d}{dx}(x^n) \cdot J_n(x) + x^n \cdot \frac{d}{dx}\big(J_n(x)\big) \rightarrow x^n \cdot J_{n-1}(x) = \frac{d}{dx}\big(x^n \cdot J_n(x)\big)$$

Conclusion,

$$\boxed{\frac{d}{dx}\big(x^n \cdot J_n(x)\big) = x^n \cdot J_{n-1}(x) - general\ formula}$$

For $n = 1$:

$$\frac{d}{dx}\big(x^1 \cdot J_1(x)\big) = x^1 \cdot J_{1-1}(x) \rightarrow \boxed{\frac{d}{dx}\big(x \cdot J_1(x)\big) = x \cdot J_0(x)}$$

1) To prove this formula, we use the representation of the Bessel function as a power series:

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$$

(More information: https://en.wikipedia.org/wiki/Bessel_function)

Then, for $\alpha = 1/2$:

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1/2+1)} \left(\frac{x}{2}\right)^{2n+1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2} \rightarrow$$

$$\left(\left(\frac{x}{2}\right)^{1/2}\right) \cdot \Bigg| J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2} \rightarrow$$

$$\left(\frac{x}{2}\right)^{1/2} \cdot J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1} \rightarrow$$

Now we need to use several properties of the gamma function:

$$\begin{cases} \Gamma(z+1) = z \cdot \Gamma(z) \\ \Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n \cdot n!}\sqrt{\pi} \\ \Gamma(n+1) = n!\,, \quad n = 0,1,2,3,4,\ldots \end{cases}$$

(More information: https://en.wikipedia.org/wiki/Gamma_function)

$$\Gamma\left(n+\frac{3}{2}\right) = \Gamma\left(\left(\frac{1}{2}+n\right)+1\right) = \left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+n\right) = \frac{2n+1}{2} \cdot \frac{(2n)!}{4^n \cdot n!}\sqrt{\pi} = \frac{2n+1}{2} \cdot \frac{(2n)!}{(2^2)^n \cdot n!}\sqrt{\pi} =$$

$$= \frac{(2n+1)!}{2^{2n+1} \cdot n!}\sqrt{\pi}$$

Then,

$$\sqrt{\frac{x}{2}}\cdot J_{1/2}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\cdot\left(\frac{(2n+1)!}{2^{2n+1}\cdot n!}\sqrt{\pi}\right)}\left(\frac{x}{2}\right)^{2n+1}\rightarrow$$

$$\sqrt{\frac{x}{2}}\cdot J_{1/2}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n!\cdot 2^{2n+1}}{n!\cdot(2n+1)!\cdot\sqrt{\pi}}\cdot\frac{x^{2n+1}}{2^{2n+1}}=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}$$

Conclusion,

$$\boxed{\sqrt{\frac{x}{2}}\cdot J_{1/2}(x)=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}}$$

As we know

$$\boxed{\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{(2n+1)!}}$$

(More information: https://en.wikipedia.org/wiki/Taylor_series)

Then,

$$\sqrt{\frac{x}{2}}\cdot J_{1/2}(x)=\frac{1}{\sqrt{\pi}}\cdot\sin(x)\Bigg|\cdot\left(\sqrt{\frac{2}{x}}\right)\rightarrow$$

$$\boxed{J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\cdot\sin(x)}$$

Commentary: The condition of this task contained a mistake, as it indicates the cosine.

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