Answer on Question #84492 – Math – Complex Analysis

Question

Using the method of residues, evaluate the following integral:

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos(\theta)}$$

Solution

Let $z = e^{i\theta}, dz = ie^{i\theta}d\theta, d\theta = \frac{dz}{iz}$, $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

The complex z describes the unit circle C_1 in the positive sense as θ varies from 0 to 2π. So, the integral becomes

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos(\theta)} = \oint_{C_1(0)} \frac{\frac{dz}{iz}}{3+2\left(\frac{z+\frac{1}{z}}{2}\right)} = -i\oint_{C_1(0)} \frac{dx}{3z+z^2+1} =$$

$$= -i(2\pi i)\sum_j \mathrm{Rez}(z^2+3z+1, z_j) = 2\pi\sum_j \mathrm{Rez}(z^2+3z+1, z_j),$$

where the sum of the residues extends over all the poles of $\frac{1}{z^2+3z+1}$ inside the unit disk.

$z^2+3z+1=0$

The roots are

$$z = \frac{-3\pm\sqrt{3^2-4(1)(1)}}{2} = \frac{-3\pm\sqrt{5}}{2}$$

$$z_1 = \frac{-3-\sqrt{5}}{2}, z_2 = \frac{-3+\sqrt{5}}{2}.$$

Only $z_2 = \frac{-3+\sqrt{5}}{2}$ is inside $C_1(0)$.

Use the formula

$\mathrm{Rez}_{z=z_0}\frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$, where z_0 is a simple zero of $h(z)$

$$\mathrm{Rez}(z^2+3z+1, z_2) = \frac{1}{2\left(\frac{-3+\sqrt{5}}{2}\right)+3} = \frac{1}{\sqrt{5}}$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos(\theta)} = \frac{2\pi}{\sqrt{5}}.$$