Answer to Question #84479 - Math – Linear Algebra

Question:

Obtain the eigenvalues and eigenvectors of the matrix: $M = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution:

The characteristic equation is $|M - \lambda I| = 0$, where the roots λ are called the eigenvalues of *M*.

$$|M - \lambda I| = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left\{(2-\lambda)^2 - 9\right\} = 0 \Rightarrow (1-\lambda)\left(4 - 4\lambda + \lambda^2 - 9\right) = 0 \Rightarrow (1-\lambda)\left(\lambda^2 - 4\lambda - 5\right) = 0.$$

$$(1-\lambda)(\lambda+1)(\lambda-5) = 0.$$

The eigenvalues are $\lambda = -1, \lambda = 1, \lambda = 5$.

The eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ are obtained by solving $(M - \lambda I)X = 0$ for each λ.

When $\lambda = -1$, $(M - \lambda I)X = \begin{pmatrix} 2+1 & 3 & 0 \\ 3 & 2+1 & 0 \\ 0 & 0 & 1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3x_1 + 3x_2 \\ 3x_1 + 3x_2 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$x_1 + x_2 = 0, x_3 = 0$. Take $x_2 = 1$.

Hence the eigenvector corresponding to $\lambda = -1$ is $X = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

When $\lambda = 1$, $(M - \lambda I)X = \begin{pmatrix} 2-1 & 3 & 0 \\ 3 & 2-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 3x_2 \\ 3x_1 + x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$x_1 = 0, x_2 = 0, x_3 = 1$.

Hence the eigenvector corresponding to $\lambda = 1$ is $X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

When $\lambda = 5$, $(M - \lambda I)X = \begin{pmatrix} 2-5 & 3 & 0 \\ 3 & 2-5 & 0 \\ 0 & 0 & 1-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3x_1 + 3x_2 \\ 3x_1 - 3x_2 \\ -4x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$x_1 - x_2 = 0, x_3 = 0$. Take $x_2 = 1$.

Hence the eigenvector corresponding to $\lambda = 5$ is $X = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.