Answer on Question #84315 – Math – Differential Equations

Question

Using method of variation of parameters, solve the equation.
d^2y/dx^2+y=Cosec x

Solution

We have the equation: y''+y=cosec(x), or $y''+y=\frac{1}{\sin(x)}$.

Solve the homogeneous equation: y''+y=0. We solve its characteristic equation: $k^2+1=0$, $k_1=i$, $k_2=-i$. The general solution to the equation will be: $y^*=C_1\cos(x)+C_2\sin(x)$.

We vary the parameters. $y^*=C_1(x)\cos(x)+C_2(x)\sin(x)$.

We make the system of equations:

$C'_1(x)\cos(x)+C_2'(x)\sin(x)=0$,

$-C'_1(x)\sin(x)+C_2'(x)\cos(x)=\frac{1}{\sin(x)}$.

From the first equation of the system we get the following: $C'_1(x)\cos(x)=-C_2'(x)\sin(x)$, $C'_1(x)=-C_2'(x)\tan(x)$. Substitute $C_1'(x)$ in the second equation of the system. We get:

$C_2'(x)\tan(x)\sin(x)+C_2'(x)\cos(x)=\frac{1}{\sin(x)}$. $C_2'(x)=\frac{\cos(x)}{\sin(x)\times((\cos(x))^2-(\sin(x))^2)}$, or $d(C_2(x))=\frac{\cos(x)dx}{\sin(x)\times((\cos(x))^2-(\sin(x))^2)}$.

We integrate the extreme expression. We get: $\int d(C_2'(x))=\int\frac{\cos(x)dx}{\sin(x)\times((\cos(x))^2-(\sin(x))^2)}$, or

$C_2(x)=\int\frac{\cos(x)dx}{\sin(x)\times((\cos(x))^2-(\sin(x))^2)}=|\sin(x)=t,\quad \cos(x)dx=dt,\quad (\cos(x))^2-(\sin(x))^2=1-2t^2|=$
$=\int\frac{dt}{t(1-2t^2)}=|\frac{A}{t}+\frac{Bt+C}{1-2t^2}\equiv\frac{1}{t(1-2t^2)}$, B-2A=0, C=0, A=1, B=2$|=\int\frac{dt}{t}+\int\frac{2tdt}{1-2t^2}=\ln|t|-\frac{1}{2}\ln|1-2t^2|+C_2^*=$
$=\ln|\sin(x)|-\frac{1}{2}\ln|1-2(\sin(x))^2|+C_2$.

Further. We have $C_1'(x)=-C_2'(x)\tan(x)=\frac{-\cos(x)}{\sin(x)\times((\cos(x))^2-(\sin(x))^2)}\times tan(x)=\frac{1}{-((\cos(x))^2-(\sin(x))^2)}=$
$=\frac{-1}{\cos(2x)}$. Then, $d(C_1'(x))=\frac{-dx}{\cos(2x)}$.

$\int d(C_1'(x))=-\int\frac{dx}{\cos(2x)}$; $C_1(x)=-\int\frac{dx}{\cos(2x)}=-\frac{1}{2}\ln|\tan(x+\frac{\pi}{4})|+C_1$. We get the general solution of the equation: $y=C_1(x)\cos(x)+C_2(x)\sin(x)=-\frac{1}{2}\times\cos(x)\times\ln|\tan(x+\frac{\pi}{4})|+C_1\cos(x)+\sin(x)\times\ln|\frac{\sin(x)}{\sqrt{(1-2(\sin(x))^2)}}|+C_2\sin(x)$.

Answer: $y=-\frac{1}{2}\cos(x)\ln|\tan(x+\frac{\pi}{4})|+C_1\cos(x)+\sin(x)\ln|\frac{\sin(x)}{\sqrt{(1-2(\sin(x))^2)}}|+C_2\sin(x)$.