Answer to Question #84219, Math / Linear Algebra

Question: Show for a square matrix, the followings are equivalent.

The columns of *A* are all vectors of length 1, and are all at right angles to each other.

$A^T = A^{-1}$.

Solution:

$\models$

Given *A* is a square matrix of order *n*.

$A = [A_1 \quad A_2 \quad A_3 \ \ldots \ A_n]$, where each $A_i = [a_{1i} \quad a_{2i} \quad a_{3i} \ \ldots \ a_{ni}]^T$ is a vector of length 1.

i.e., $\sqrt{a_{1i}^2 + a_{2i}^2 + \ldots + a_{ni}^2} = 1 \qquad \Rightarrow a_{1i}^2 + a_{2i}^2 + \ldots + a_{ni}^2 = 1$.

$A_i^2 = 1$.

Also the columns of *A* are all at right angles to each other. i.e., $A_i \cdot A_j = 0$.

i.e., $A_i A_j^T = 0, \ \forall i \neq j \qquad \Rightarrow a_{1i}a_{j1} + a_{2i}a_{j2} + \ldots + a_{ni}a_{jn} = 0, \ \forall i \neq j$.

Now $AA^T = \begin{bmatrix} A_1^2 & A_1A_2^T & A_1A_3^T & A_1A_n^T \\ A_2A_1^T & A_2^2 & A_2A_3^T & \ldots A_2A_n^T \\ A_3A_1^T & A_3A_2^T & A_3^2 & A_3A_n^T \\ . & & & \\ . & & & \\ A_nA_1^T & A_nA_2^T & A_nA_3^T & \ldots \ A_n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 \\ . & & & \\ . & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$.

Therefore, $A^T = A^{-1}$.

$\Leftarrow$

Given $A^T = A^{-1}$.

i.e., $AA^T = I \quad \Rightarrow \begin{bmatrix} A_1^2 & A_1A_2^T & A_1A_3^T & A_1A_n^T \\ A_2A_1^T & A_2^2 & A_2A_3^T & \ldots A_2A_n^T \\ A_3A_1^T & A_3A_2^T & A_3^2 & A_3A_n^T \\ . & & & \\ . & & & \\ A_nA_1^T & A_nA_2^T & A_nA_3^T \ldots & A_n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 \\ . & & & \\ . & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This implies $A_i^2 = 1, \forall i = 1, 2, \ldots, n$.

Thus each vector A_i is of length 1.

And $A_i A_j^T = 0, \ \forall i \neq j$, which implies that $A_i \cdot A_j = 0, \ \forall i \neq j$.

Thus the columns of *A* are all at right angles to each other.

Thus the equivalence of the two statements is proved.