Answer on Question #84158 – Math – Differential Equations

Question

Find the general solution of the differential equation

(4d"y/dx")+4(dy/dx)+65y=65x^2 +8x+73

Show that, whatever the initial conditions, y/x^2--->1 as x reaches infinity

Solution

So, we have a second-order differential equation with the right-hand side in the form of a polynomial with constant coefficients, which we write as: $4y'' + 4y' + 65y = 65x^2 + 8x + 73$. We find the general solution of a linear homogeneous equation corresponding to this equation, that is, the equation: $4y'' + 4y' + 65y = 0$. Find the roots of its characteristic equation: $4k^2 + 4k + 65 = 0$. $k_{1,2} = \frac{-4\pm\sqrt{4^2-4\times4\times65}}{2\times4} = \frac{-4\pm\sqrt{-1024}}{8} = \frac{-4\pm32i}{8} = -\frac{1}{2} \pm 4i$. Then the general solution of a linear homogeneous equation is written in the form: $y_{gh}(x) = e^{\frac{-x}{2}}(C_1\cos(4x) + {}$ $+C_2\sin(4x))$.

$y'_p(x) = 2Ax + B$; $y''_p(x) = 2A$. Now we substitute the derived derivatives and the function itself into our equation, in which, however, we note that the equal sign will mean the sign of the identical equality, that is, the values of these will be respectively equal. Then we get the following identity equality: $8A + 8Ax + 4B + 65 \times (Ax^2 + Bx + C) = 65x^2 + 8x + 73$ or $65Ax^2 + x(65B + 8A) + (8A + 4B + 65C) = 65x^2 + 8x + 73$. From this identical equality we get that: $65A = 65, and\ A = 1; 65B + 8 \times 1 = 8, and\ B = 0; 8 + 0 + 65C = 73, and\ C == 1$. Thus, a particular solution of a differential equation will be: $y_p(x) = x^2 + 1$. And the general solution of the inhomogeneous equation will be: $y_{gi}(x) = y_{gh}(x) + y_p(x) == e^{\frac{-x}{2}}(C_1\cos(4x) + C_2\sin(4x)) + x^2 + 1$.

We turn to the second part of the problem. We need to show that at $x \to \infty$ expression $\frac{y_{gi}(x)}{x^2} \to 1$ under any initial conditions, that is, in essence, we find the limit: $\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))+x^2+1}{x^2}$=$\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))}{x^2}$+$\lim\limits_{x\to\infty}\frac{x^2}{x^2}$+$\lim\limits_{x\to\infty}\frac{1}{x^2}$=

$=\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))}{x^2}$+$\lim\limits_{x\to\infty} 1$+$\frac{1}{\infty}$=$\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))}{x^2}$+1+0=

$=\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))}{x^2}$+1=$\lim\limits_{x\to\infty}\frac{C_1\cos(4x)+C_2\sin(4x)}{x^2\times e^{\frac{x}{2}}}$+1. Analyzing the limit $\lim\limits_{x\to\infty}\frac{C_1\cos(4x)+C_2\sin(4x)}{x^2\times e^{\frac{x}{2}}}$, we see that in the numerator there are those functions that are bounded by -1 from below and by +1 from above with an infinite increase of x, and in the denominator there are functions that will increase infinitely with infinite growth of x, then $\lim\limits_{x\to\infty}\frac{C_1\cos(4x)+C_2\sin(4x)}{x^2e^{\frac{x}{2}}}$=0, and $\lim\limits_{x\to\infty}\frac{e^{\frac{-x}{2}}(C_1\cos(4x)+C_2\sin(4x))+x^2+1}{x^2}$=1, what was required to show.

Answer: $y_{gi}(x) = e^{\frac{-x}{2}}(C_1 \cos(4x) + C_2 \sin(4x)) + x^2 + 1.$