Answer to Question #84074, Math / Real Analysis

Question: Every subsequence of the sequence $\left(\frac{1}{n^2}\right)$ is convergent.

Solution:

First we prove that the sequence $\left(\frac{1}{n^2}\right)$ is convergent. Then we shall show that every subsequence of a convergent sequence converges.

Let $\varepsilon > 0$ be given. By Archimedean property, there exists a $N \in \mathbb{N}$ such that $\frac{1}{N^2} < \varepsilon$.

For all $n \geq N$, $\left(\frac{1}{n^2} - 0\right) = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon$.

Thus the sequence $\left(\frac{1}{n^2}\right)$ converges to 0.

Now let (b_n) be any subsequence of the sequence (a_n) where $a_n = \frac{1}{n^2}$.

Let $\varepsilon > 0$ be given. For $n \geq N$, $b_n = a_m$ for some $m \geq n \geq N$.

$|b_n - 0| = |a_m - 0| < \varepsilon$ for all $n \geq N$.

Thus the subsequence (b_n) is convergent.

Hence every subsequence of a convergent sequence is convergent.