Answer to Question #84065 – Math – Differential Equations

Given differential equation

$$y = x\left(p^2 - 2p + 2\right)$$

$$\Rightarrow p^2 - 2p + 2 = y/x \Rightarrow p^2 - 2p + 1 = \frac{y}{x} - 1$$

$$\Rightarrow (p-1)^2 = \frac{y}{x} - 1 \Rightarrow (p-1) = \pm\sqrt{\frac{y}{x} - 1} \Rightarrow p = 1 \pm \sqrt{\frac{y}{x} - 1}$$

$$\Rightarrow \frac{dy}{dx} = 1 \pm \sqrt{\frac{y}{x} - 1}.........(1)$$

$$let\ \frac{y}{x} = t \Rightarrow y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$putting\ in (1)\ t + x\frac{dt}{dx} = 1 \pm \sqrt{t-1}$$

$$\Rightarrow x\frac{dt}{dx} = 1 - t \pm \sqrt{t-1} \Rightarrow x\frac{dt}{dx} = 1 - t \pm \sqrt{t-1}$$

$$\Rightarrow \frac{dt}{1 - t \pm \sqrt{t-1}} = \frac{dx}{x} \Rightarrow \frac{dt}{\sqrt{t-1}\left(\pm 1 - \sqrt{t-1}\right)} = \frac{dx}{x}$$

Integrating $\int \frac{dt}{\sqrt{t-1}\left(\pm 1 - \sqrt{t-1}\right)} = \int \frac{dx}{x} + \ln c = \ln x + \ln c$ ....(2)

$$let \left(\pm 1 - \sqrt{t-1}\right) = u \Rightarrow -\frac{dt}{2\sqrt{t-1}} = du \Rightarrow \frac{dt}{\sqrt{t-1}} = -2du$$

Now (2) becomes $\int \frac{-2du}{u} = \ln x + \ln c \Rightarrow -2\ln u = \ln cx \Rightarrow u^{-2} = cx \Rightarrow u^2 = \frac{1}{cx}$

$$\Rightarrow \left(\pm 1 - \sqrt{t-1}\right)^2 = \frac{1}{cx} \Rightarrow \pm 1 - \sqrt{t-1} = \pm\frac{1}{\sqrt{cx}} \Rightarrow \sqrt{t-1} = \pm 1 \pm \frac{1}{\sqrt{cx}}$$

$$\Rightarrow \sqrt{\frac{y}{x} - 1} = \pm 1 \pm \frac{1}{\sqrt{cx}} \Rightarrow \frac{y}{x} - 1 = \left(\pm 1 \pm \frac{1}{\sqrt{cx}}\right)^2 \Rightarrow \frac{y}{x} = 1 + \left(\pm 1 \pm \frac{1}{\sqrt{cx}}\right)^2 \Rightarrow y = x\left[1 + \left(\pm 1 \pm \frac{1}{\sqrt{cx}}\right)^2\right]$$

hence there are four solutions

$$y = x\left[1 + \left(1 + \frac{1}{\sqrt{cx}}\right)^2\right],\ y = x\left[1 + \left(1 - \frac{1}{\sqrt{cx}}\right)^2\right],\ y = x\left[1 + \left(-1 + \frac{1}{\sqrt{cx}}\right)^2\right],\ y = x\left[1 + \left(-1 - \frac{1}{\sqrt{cx}}\right)^2\right]$$

Equivalently we have two solutions $y = x\left[1+\left(1+\frac{1}{\sqrt{cx}}\right)^2\right]$, $y = x\left[1+\left(1-\frac{1}{\sqrt{cx}}\right)^2\right]$