Answer on Question #84013 – Math – Other

Question

Q1. A system of linear equations is shown below.

$$\begin{cases} 2x + 5y + 6z = 4 \\ x - 6y + 2z = 9 \\ 3x - 2y + 4z = 8 \end{cases}$$

(a) Write down the augmented matrix for the above system of equations.
(b) Determine whether the system of equations has unique solution.
(c) Use Gaussian elimination method to solve the above system of equations.

Solution

(a)
Augmented matrix

$$A = \begin{pmatrix} 2 & 5 & 6 & 4 \\ 1 & -6 & 2 & 9 \\ 3 & -2 & 4 & 8 \end{pmatrix}$$

(b)

$$\begin{vmatrix} 2 & 5 & 6 \\ 1 & -6 & 2 \\ 3 & -2 & 4 \end{vmatrix} = 2\begin{vmatrix} -6 & 2 \\ -2 & 4 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + 6\begin{vmatrix} 1 & -6 \\ 3 & -2 \end{vmatrix} =$$

$$= 2\big(-6(4) - 2(-2)\big) - 5\big(1(4) - 2(3)\big) + 6\big(1(-2) - (-6)(3)\big) = 66 \neq 0$$

Therefore, the system of equations has the unique solution.

(c)

$$\begin{pmatrix} 2 & 5 & 6 & 4 \\ 1 & -6 & 2 & 9 \\ 3 & -2 & 4 & 8 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 1 & -6 & 2 & 9 \\ 3 & -2 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 1 & -6 & 2 & 9 \\ 3 & -2 & 4 & 8 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 0 & -17/2 & -1 & 7 \\ 3 & -2 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 0 & -17/2 & -1 & 7 \\ 3 & -2 & 4 & 8 \end{pmatrix} \xrightarrow{R_3 - (3)R_1} \begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 0 & -17/2 & -1 & 7 \\ 0 & -19/2 & -5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 0 & -17/2 & -1 & 7 \\ 0 & -19/2 & -5 & 2 \end{pmatrix} \xrightarrow{-\left(\frac{2}{17}\right)R_2} \begin{pmatrix} 1 & 5/2 & 3 & 2 \\ 0 & 1 & 2/17 & -14/17 \\ 0 & -19/2 & -5 & 2 \end{pmatrix}$$

$$\begin{pmatrix}1 & 5/2 & 3 & 2\\ 0 & 1 & 2/17 & -14/17\\ 0 & -19/2 & -5 & 2\end{pmatrix}\xrightarrow{R_1-\left(\frac{5}{2}\right)R_2}\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & -19/2 & -5 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & -19/2 & -5 & 2\end{pmatrix}\xrightarrow{R_3+\left(\frac{19}{2}\right)R_2}\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & -66/17 & -99/17\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & -66/17 & -99/17\end{pmatrix}\xrightarrow{\left(-\frac{17}{66}\right)R_3}\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & 1 & 3/2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 46/17 & 69/17\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & 1 & 3/2\end{pmatrix}\xrightarrow{R_1-\left(\frac{46}{17}\right)R_3}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & 1 & 3/2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 2/17 & -14/17\\ 0 & 0 & 1 & 3/2\end{pmatrix}\xrightarrow{R_2-\left(\frac{2}{17}\right)R_3}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 3/2\end{pmatrix}$$

$$\begin{cases} x = 0\\ y = -1\\ z = 3/2\end{cases}$$

Answer: $\left(0, -1, \frac{3}{2}\right)$.

Question

Q2. A student wants to study the dynamic characteristics of a low-rise building under lateral loading. The student models the building by a single lumped mass model. After considering damping, stiffness and characteristics of loading, the student comes up with the following differential equation:

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 8x = \sin t$$

where $x(t)$ is the time-dependent function for the lateral displacement of the building.

(a) Find the complementary function of the lateral displacement.

(b) Find the particular integral of the resulting displacement.

(c) Find the particular solution of the differential equation, given that $x(0) = 0$ and $x'(0) = 1$.

Solution

(a)

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 8x = 0$$

Auxiliary (characteristic) equation

$$r^2 - 6r + 8 = 0$$
$$(r-2)(r-4) = 0$$

$r_1 = 2, r_2 = 4$

Complementary function

$$x_{cf} = c_1 e^{2t} + c_2 e^{4t}$$

(b)

$x_p = A\cos t + B\sin t$

$x_p' = -A\sin t + B\cos t$

$x_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t + 6A\sin t - 6B\cos t + 8A\cos t + 8B\sin t = \sin t$

$$\begin{cases} 7A - 6B = 0 \\ 7B + 6A = 1 \end{cases} \Rightarrow \begin{cases} A = \dfrac{6}{85} \\ B = \dfrac{7}{85} \end{cases}$$

$$x_p = \frac{6}{85}\cos t + \frac{7}{85}\sin t$$

(c)

$$x(t) = x_{cf} + x_p = c_1 e^{2t} + c_2 e^{4t} + \frac{6}{85}\cos t + \frac{7}{85}\sin t$$

$$x(0) = c_1 + c_2 + \frac{6}{85} = 0$$

$$x'(t) = 2c_1e^{2t} + 4c_2e^{4t} - \frac{6}{85}\sin t + \frac{7}{85}\cos t$$

$$x'(0) = 2c_1 + 4c_2 + \frac{7}{85} = 1$$

$$\begin{cases} c_1 + c_2 + \frac{6}{85} = 0 \\ 2c_1 + 4c_2 + \frac{7}{85} = 1 \end{cases} => \begin{cases} c_2 = -\frac{6}{85} - c_1 \\ 2c_1 - 4c_1 - \frac{24}{85} + \frac{7}{85} = 1 \end{cases} => \begin{cases} c_1 = -\frac{3}{5} \\ c_2 = \frac{45}{85} \end{cases}$$

$$x^*(t) = -\frac{3}{5}e^{2t} + \frac{45}{85}e^{4t} + \frac{6}{85}\cos t + \frac{7}{85}\sin t.$$