Answer on Question #83959 – Math – Analytic Geometry

Question

$$\frac{a^2+b^2}{ab+1} = 4$$

Give a equation to solve the value of a and b.
a=? & b?

Solution

$$a^2+b^2-4ab-4=0$$

$$a_{11}a^2+2a_{12}ab+2a_{13}a+a_{22}b^2+2a_{23}b+a_{33}=0$$

$$a_{11}=1$$

$$a_{12}=-2$$

$$a_{13}=0$$

$$a_{22}=1$$

$$a_{23}=0$$

$$a_{33}=-4$$

Then:

$$I_1=a_{11}+a_{22}=1+1=2$$

$$I_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}=\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}=1-4=-3$$

$$I_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}=\begin{vmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -4 \end{vmatrix}=-4+2\cdot 8=12$$

Since $I_2<0$ and $I_3\neq 0$, then we have equation of hyperbola.

So:

$$I(\lambda)=\begin{vmatrix} -\lambda+1 & -2 \\ -2 & -\lambda+1 \end{vmatrix}=(-\lambda+1)^2-4=\lambda^2-2\lambda+1-4=\lambda^2-2\lambda-3=0$$

$$\lambda_1 = 3$$

$$\lambda_2 = -1$$

For $\lambda_1 = 3$ one gets $a_2 = -a_1$, hence $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2+(-1)^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

For $\lambda_2 = -1$ one gets $a_2 = a_1$, hence $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{1^2+1^2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Let $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then the rotation is $\begin{pmatrix} a \\ b \end{pmatrix} = R\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}} \\ -\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}} \end{pmatrix}$,

hence $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = R^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \end{pmatrix}$

Substituting $a = \frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}$, $b = -\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}$ into the equation $a^2 + b^2 - 4ab - 4 = 0$ one gets

$$\left(\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}\right)^2 + \left(-\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}\right)^2 - 4\left(\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}\right)\left(-\frac{\tilde{a}}{\sqrt{2}} + \frac{\tilde{b}}{\sqrt{2}}\right) - 4 = 0$$

$$\frac{1}{2}(\tilde{a})^2 + \tilde{a}\tilde{b} + \frac{1}{2}\left(\tilde{b}\right)^2 + \frac{1}{2}(\tilde{a})^2 - \tilde{a}\tilde{b} + \frac{1}{2}\left(\tilde{b}\right)^2 - 2\left(\left(\tilde{b}\right)^2 - (\tilde{a})^2\right) - 4 = 0$$

$$3\tilde{a}^2 - \tilde{b}^2 - 4 = 0$$

One gets an equation of hyperbola:

$$\frac{\tilde{a}^2}{\frac{4}{3}} - \frac{\tilde{b}^2}{4} = 1$$

The canonical form of the equation:

$$\tilde{a}^2\lambda_1 + \tilde{b}^2\lambda_2 + \frac{I_3}{I_2} = 0$$

$$3\tilde{a}^2 - \tilde{b}^2 - 4 = 0$$

$$\frac{\tilde{a}^2}{\frac{4}{3}} - \frac{\tilde{b}^2}{4} = 1$$

Answer: $\frac{\tilde{a}^2}{\frac{4}{3}} - \frac{\tilde{b}^2}{4} = 1$