

Answer on Question # 83830, Math / Functional Analysis

Question 1. Find a measure (of ?) dense subset (?) in \mathbb{R}^2 or

Find a measurable dense subset in \mathbb{R}^2 ?

Solution. \mathbb{Q}^2 is dense subset of \mathbb{R}^2 and $m(\mathbb{Q}^2) = 0$.

We construct an example of a dense open set of positive measure with the help of some subset \mathcal{D} . Let \mathcal{D} be a dense, open subset of \mathbb{R}^2 , whose Lebesgue measure is positive and finite. Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $f(x, y) := \frac{\epsilon}{m(\mathcal{D})}(x, y)$, $\epsilon > 0$. Since f is linear, $m(f(\mathcal{D})) = \frac{\epsilon}{m(\mathcal{D})}m(\mathcal{D}) = \epsilon$. Since f is continuous, $f(\mathcal{D})$ is dense in \mathbb{R}^2 . Since f is invertible, and its inverse is again continuous, $f(\mathcal{D})$ is open.

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