# Answer on Question #83664 – Math – Calculus

## Question

Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions x+y+z=1 and xyz+1=0 by using Lagrange's multiplier method

## Solution

Let $f(x,y,z) = x^2 + y^2 + z^2, g(x,y,z) = x + y + z - 1 = 0, h(x,y,z) = xyz + 1 = 0$. The gradient vectors for $f, g, h$ are:

$$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle, \qquad \nabla g(x,y,z) = \langle 1,1,1 \rangle, \qquad \nabla h(x,y,z) = (yz, xz, xy).$$

According to Lagrange's multiplier method we find all values of $x, y, z, \lambda, \mu$ such that

$$\nabla L(x,y,z) = \nabla f(x,y,z) + \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) = 0,$$
$$g(x,y,z) = 0,$$
$$h(x,y,z) = 0.$$

Where $\lambda, \mu$ are the Lagrange's multipliers.
So consider the system:

$$2x + \lambda + \mu yz = 0,$$
$$2y + \lambda + \mu xz = 0,$$
$$2z + \lambda + \mu xy = 0,$$
$$x + y + z - 1 = 0,$$
$$xyz + 1 = 0.$$

Choose the equations:

$$\lambda + \mu yz = -2x,$$
$$\lambda + \mu xz = -2y.$$

Find $\lambda, \mu$, using Cramer's rules:

$$\Delta = \begin{vmatrix} 1 & yz \\ 1 & xz \end{vmatrix} = xz - yz = z(x - y),$$
$$\Delta_\lambda = \begin{vmatrix} -2x & yz \\ -2y & xz \end{vmatrix} = -2x^2 z + 2y^2 z = -2z(x^2 - y^2),$$
$$\Delta_\mu = \begin{vmatrix} 1 & -2x \\ 1 & -2y \end{vmatrix} = -2y + 2x = 2(x - y).$$

Consider two cases, when $\Delta = 0$, and $\Delta \neq 0$.
If $\Delta = 0$, so $z = 0$ or $x - y = 0$. But $z = 0$ don't satisfy the equation $xyz + 1 = 0$. So we have $z \neq 0$.
Consider when $x - y = 0$, or $x = y$. So for the equations $x + y + z - 1 = 0,\ xyz + 1 = 0$ we have:

$$2y + z - 1 = 0, \qquad z = 1 - 2y,$$
$$y^2 z + 1 = 0, \qquad y^2(1 - 2y) + 1 = 0.$$

Solve the equation:

$$y^2 - 2y^3 + 1 = 0,$$
$$y^3 - y^2 + y^3 - 1 = 0,$$
$$y^2(y-1) + (y-1)(y^2 + y + 1) = 0,$$
$$(y-1)(2y^2 + y + 1) = 0,$$
$$y = 1 \text{ or } 2y^2 + y + 1 = 0.$$

So we have $y = x = 1$, $z = -1$. Substitute these values in the equations

$$\lambda + \mu yz = -2x,$$
$$\lambda + \mu xy = -2z.$$

We find $\lambda, \mu$:

$$\lambda - \mu = -2,$$
$$\lambda + \mu = 2.$$

So $\lambda = 0, \mu = 2$.

So we have the first point $M_1 = (1,1,-1)$ and $\lambda_1 = 0,\ \mu_1 = 2$.

The equation $2y^2 + y + 1 = 0$ does not have real roots ($D < 0$).

If $\Delta \neq 0$, we have:

$$\lambda = \frac{\Delta_\lambda}{\Delta} = -\frac{2z(x^2 - y^2)}{z(x-y)} = -2(x+y),$$
$$\mu = \frac{\Delta_\mu}{\Delta} = \frac{2(x-y)}{z(x-y)} = \frac{2}{z}$$

Substitute $\lambda = -2(x+y), \mu = \frac{2}{z}$ in the third equation $\lambda + \mu xy = -2z$:

$$-2(x+y) + \frac{2}{z}xy == -2z,$$
$$z - x - y + \frac{1}{z}xy = 0.$$

Use the equations:

$x + y + z - 1 = 0, \quad x + y = 1 - z,$

$xyz + 1 = 0, \qquad xy = -\frac{1}{z}.$

So write the equation $z - x - y + \frac{1}{z}xy = 0$:

$$2z - \frac{1}{z^2} - 1 = 0,$$

We know $z \neq 0$, so we have

$$2z^3 - z^2 - 1 = 0,$$
$$(z-1)(z^2 + z + 1) = 0,$$
$$z = 1 \text{ or } z^2 + z + 1 = 0 \text{ (has not real roots).}$$

Substitute $z = 1$ in the equations $x + y = 1 - z$,

$$x + y = 1 - z,$$
$$xy = -\frac{1}{z}.$$

We have

$$x + y = 0,$$
$$xy = -1.$$

We have the solution: $x = -1, y = 1$ and $x = 1, y = -1$. And $\lambda = 0, \mu = 2$.

So we have the points $M_2 = (-1,1,1)$, $M_3 = (1,-1,1)$ and and $\lambda_2 = \lambda_3 = 0,\ \mu_2 = \mu_3 = 2$.

For each point $f(M_1) = f(M_2) = f(M_3) = 3$. Define if this value is maximum or minimum value. Write

$$L(x,y,z) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$$
$$L(x,y,z) = x^2 + y^2 + z^2 + \lambda(x+y+z-1) + \mu(xyz+1).$$

Then find the matrix:

$$\begin{pmatrix} 0 & 0 & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial y} \\ 0 & 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial z} \\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial^2 L}{\partial x \partial z} & \frac{\partial^2 L}{\partial y \partial z} & \frac{\partial^2 L}{\partial z^2} \end{pmatrix}$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 1, \frac{\partial h}{\partial x} = yz\,, \frac{\partial h}{\partial y} = xz\,, \frac{\partial h}{\partial z} = xy.$$

$$\frac{\partial L}{\partial x} = 2x + \lambda + \mu yz, \frac{\partial L}{\partial y} = 2y + \lambda + \mu xz, \frac{\partial L}{\partial z} = 2z + \lambda + \mu xy$$

$$\frac{\partial^2 L}{\partial x^2} = 2, \frac{\partial^2 L}{\partial y^2} = 2, \frac{\partial^2 L}{\partial x^2} = 2, \frac{\partial^2 L}{\partial z^2} = 2, \frac{\partial^2 L}{\partial x \partial y} = \mu z, \frac{\partial^2 L}{\partial x \partial z} = \mu y, \frac{\partial^2 L}{\partial y \partial z} = \mu x.$$

So our matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & yz & xz & xy \\ 1 & yz & 2 & \mu z & \mu y \\ 1 & xz & \mu z & 2 & \mu x \\ 1 & xy & \mu y & \mu x & 2 \end{pmatrix}$$

For the point $M_1 = (1,1,-1)$ and $\lambda_1 = 0,\ \mu_1 = 2$ this matrix is:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 2 & -2 & 2 \\ 1 & -1 & -2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

If the signs of the corner minors $H_{2m+1,}, H_{2m+2}, \ldots H_{m+n}$ of the matrix $A$ equal the signs of $(-1)^m$, then the stationary point is minima of the objective function. If the signs of the corner minors $H_{2m+1,}, H_{2m+2}, \ldots H_{m+n}$ of the matrix $A$ interchange, besides the sign of $H_{2m+1,}$ equals the sign of $(-1)^{m+1}$, then the stationary point is maxima of the objective function. Here $m$ is the number of the conditions and $n$ is the number of the variables. In our case $m = 2, n = 3$. So we define the sign $H_{2m+1,} = H_{m+n} = H_5 = \det A$.

$$\det A = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 2 & -2 & 2 \\ 1 & -1 & -2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{vmatrix} = R_4 \to R_4 - R_3, R_5 \to R_5 - R_3 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 2 & -2 & 2 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 2 & 0 & 4 & 0 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & -1 & 2 & -2 & 2 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{vmatrix} = R_4 \to R_4 + 4R_3, R_5 \to R_5 + R_3 = \begin{vmatrix} 1 & -1 & 2 & -2 & 2 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 8 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot 8 \cdot 2 = 32 > 0$$

The sign $H_5 > 0$, and the sign $(-1)^2 > 0$, so for $M_1 = (1,1,-1)$ the function $f(x, y, z)$ has minimum value $f(1,1,-1) = 3$.

Thus, we have that the minimum value of $f$ is 3. According to the given conditions the function $f$ does not have the maximum value.

**Answer:** the minimum value of $f$ is 3; the maximum value of $f$ does not exist.