Answer on Question #83505 – Math – Linear Algebra

Question

Check whether or not the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$$

is diagonalizable. If it is, find a matrix P, and a matrix D such that $P^{-1}AP = D$. If A is not diagonalizable find Adj A.

Solution

Matrix A (3×3) is diagonalizable if and only if there is a basis of R^3 consisting of eigenvectors of A. So let's find the eigenvalues and eigenvectors for matrix A.

To define the eigenvalues, we find the values of $\boldsymbol{\lambda}$ which satisfy the characteristic equation of the matrix A:

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 identity matrix.
Form the matrix $A - \lambda I$:

$$A - \lambda I == \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -2-\lambda & 2 \\ 0 & -2 & -3-\lambda \end{bmatrix}.$$

Calculate $\det(A - \lambda I)$. We define the determinant of a 3 × 3 matrix by the rule:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

$$\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 0 & -3-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & -2-\lambda \\ 0 & -2 \end{vmatrix} =$$
$$= (1-\lambda)\big((-2-\lambda)(-3-\lambda) - (-2)2\big) = (1-\lambda)(6 + 5\lambda + \lambda^2 + 4)$$
$$= (1-\lambda)(\lambda^2 + 5\lambda + 10)$$

Then find solutions of the characteristic equation $\det(A - \lambda I) = 0$, i.e. solve the equation:

$$(1-\lambda)(\lambda^2 + 5\lambda + 10) = 0$$

$$1 - \lambda = 0 \text{ or } \lambda^2 + 5\lambda + 10 = 0$$

$$\lambda_1 = 1$$

The quadratic equation $\lambda^2 + 5\lambda + 10 = 0$ has not any real roots, because

$$\Delta = b^2 - 4ac = 25 - 40 = 15 < 0.$$

So we can find only one eigenvector for A. But any basis for R^3 consists of three vectors. Therefore, there is no eigenbasis for A, and so the matrix A is not diagonalizable.

Find Adj A. First, calculate the cofactors of each element.

$$c_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 2 \\ -2 & -3 \end{vmatrix} = 6 + 4 = 10, c_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 0 & -3 \end{vmatrix} = 0,$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ 0 & -2 \end{vmatrix} = 0, \quad c_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} = -1(-3+2) = 1,$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = -3, \qquad c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -1(-2) = 2,$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = 2 + 2 = 4, c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -1(2) = -2,$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -2.$$

Write the cofactor matrix C:

$$C = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & 2 \\ 4 & -2 & -2 \end{bmatrix}, \ adj\ A = C^T = \begin{bmatrix} 10 & 1 & 4 \\ 0 & -3 & -2 \\ 0 & 2 & -2 \end{bmatrix}.$$

Answer: *A* is not diagonalizable; $adj\ A = \begin{bmatrix} 10 & 1 & 4 \\ 0 & -3 & -2 \\ 0 & 2 & -2 \end{bmatrix}.$