Answer on Question #83328 – Math – Differential Equations

Question

Find a power series solution in powers of x. Show the details.

$$y'' + y' + yx^2 = 0$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute into the equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$n = 0 \;\Rightarrow\; a_0 = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + a_1 = 0 \;\; for\; n = 0$$

$$6a_3 x + 2a_2 x = 0 \; for\; n = 1$$

$$4a_2 + a_1 + 6a_3 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-2}]x^n = 0$$

$$n = 2,3,4,\dots \Rightarrow (n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-2} = 0$$

Thus, we have

$$a_2 = -\frac{a_1}{2}$$

$$a_3 = -\frac{a_2}{3} = \frac{a_1}{6}$$

For $n = 2$:

$$4 \cdot 3a_4 + 3a_3 = 0 \ \Rightarrow \ a_4 = -\frac{a_3}{4} = -\frac{a_1}{24}$$

For $n = 3$:

$$5 \cdot 4a_5 + 4a_4 + a_1 = 0 \ \Rightarrow \ a_5 = -\frac{4a_4 + a_1}{20} = -\frac{a_1}{24}$$

For $n = 4$:

$$6 \cdot 5a_6 + 5a_5 + a_2 = 0 \ \Rightarrow \ a_6 = -\frac{5a_5 + a_2}{30} = -\frac{-\frac{5a_1}{24} - \frac{a_1}{2}}{30} = \frac{17a_1}{24 \cdot 30}$$

For $n = 5$:

$$7 \cdot 6a_7 + 6a_6 + a_3 = 0 \ \Rightarrow \ a_7 = -\frac{6a_6 + a_3}{42} = -\frac{\frac{17a_1}{4 \cdot 30} + \frac{a_1}{6}}{42} = -\frac{37a_1}{120 \cdot 42}$$

For $n = 6$:

$$8 \cdot 7a_8 + 7a_7 + a_4 = 0 \ \Rightarrow \ a_8 = -\frac{7a_7 + a_4}{56} = -\frac{-\frac{37a_1}{120 \cdot 6} + \frac{17a_1}{24 \cdot 30}}{56} = \frac{20a_1}{56 \cdot 720}$$

Answer:

$$y = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{24} + \frac{17x^6}{720} - \frac{37x^7}{5040} + \frac{x^8}{2016} + \cdots\right)$$