Answer on Question #83181– Math – Differential Equations

Question

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=4\cos t+2\sin t, x(0)=0; x(0)=3.$$

Solution

Solve second order linear nonhomogeneous differential equation with constant coefficients.

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=4\cos t+2\sin\ t.$$

The general solution of a nonhomogeneous equation is the sum of the general solution $x_0(t)$ of the related homogeneous equation and a particular solution $x_1(t)$ of the nonhomogeneous equation:

$$x(t)=x_0(t)+x_1(t).$$

First we solve the related homogeneous equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=0.$$

Find the roots of the corresponding characteristic equation:

$$k^2+2k+2=0,$$

$$D=2^2-4\cdot 2=-4=4i^2, \text{ so } k_{1,2}=\frac{-2\pm\sqrt{4i}}{2}=-1\pm i.$$

If the roots of the characteristic equation are the complex numbers $k_{1,2}=\alpha\pm i\beta$, then the general solution of the homogeneous equation $x(t)=e^{\alpha t}(C_1\cos\beta t+C_2\sin\beta t)$. In our case we have:

$$x_0(t)=e^{-t}(C_1\cos t+C_2\sin t).$$

Now find the particular solution $x_1(\mathrm{t})$ of the nonhomogeneous equation. The right side of a nonhomogeneous differential equation is the combination of trigonometric functions. In this case, it's more convenient to look for a solution using the method of undetermined coefficients. Write the particular solution $x_1(\mathrm{t})$ in the form:

$x_1(t)=A\cos t+B\sin t$, where A, B are the undetermined coefficients.

Find $\frac{dx_1}{dt},\frac{d^2x_1}{dt^2}$:

$$\frac{dx_1}{dt}=-A\sin t+B\cos t,$$

$$\frac{d^2x_1}{dt^2} = -A\cos t - B\sin t.$$

Substitute them back into the original differential equation:

$$-A\cos t - B\sin t - 2A\sin t + 2B\cos t + 2A\cos t + 2B\sin t = 4\cos t + 2\sin\ t,$$

$$(A + 2B)\cos t + (B - 2A)\sin t = 4\cos t + 2\sin\ t.$$

Therefore we can write the following system of equations to determine the coefficients A, B:

$$\begin{cases} A + 2B = 4, \\ B - 2A = 2, \end{cases} \begin{cases} A + 4A + 4 = 4, \\ B = 2A + 2, \end{cases} \begin{cases} A = 0, \\ B = 2. \end{cases}$$

Thus, the particular solution has the form:

$$x_1(t) = 2\sin t.$$

Respectively, the general solution of the original nonhomogeneous equation is written as:

$$x(t) = e^{-t}(C_1\cos t + C_2\sin t) + 2\sin t.$$

Find the solution with the initial conditions $x(0) = 0; x(0) = 3$.

$$x(0) = e^0(C_1\cos 0 + C_2\sin 0) + 2\sin 0 = C_1.$$

So we have that $C_1 = 0$ and $C_1 = 3$. But it's impossible and the solution with the initial conditions $x(0) = 0; x(0) = 3$ doesn't exist.

But if the initial conditions $x(0) = 0; x'(0) = 3$ should be considered we can find the solution.

First find $x'(t)$:

$$x'(t) = -e^{-t}(C_1\cos t + C_2\sin t) + e^{-t}(-C_1\sin t + C_2\cos t) + 2\cos t,$$

$$x'(0) = -e^0(C_1\cos 0 + C_2\sin 0) + e^0(-C_1\sin 0 + C_2\cos 0) + 2\cos 0 = -C_1 + C_2 + 2,$$

So

$$\begin{cases} C_1 = 0, \\ -C_1 + C_2 + 2 = 3, \end{cases} \begin{cases} C_1 = 0, \\ C_2 = 1. \end{cases}$$

Therefore, the solution of the initial-value problem is:

$$x(t) = e^{-t}(0 \cdot \cos t + 1 \cdot \sin t) + 2\sin t = e^{-t}\sin t + 2\sin t.$$

Answer: The solution doesn't exist for the initial conditions $x(0) = 0; x(0) = 3$.

For the initial conditions $x(0) = 0; x'(0) = 3$ the solution of the initial-value problem is

$$x(t) = e^{-t} \sin t + 2 \sin t.$$