Answer on Question # 83120, Math / Abstract Algebra

Question 1. *Verify that $k[[X]]$ is a local ring, where k is a field.*

Solution. Let k be a field and let X be an indeterminate; denote by $k[[X]]$ the set of all formal expressions

$$\sum_{n=0}^{\infty} a_n X^n,\ a_n \in k.$$

More precisely, $k[[X]]$ as a k-vector space is the direct product of a denumerable number of copies of k indexed by the set of monomials X^n, $n \geqslant 0$. $k[[X]]$ is made into a ring by defining addition and multiplication exactly as for polynomials except there is no restriction that the result must have all but a finite number of coefficients 0. In the formula for the product

$$\left(\sum_i a_i X^i\right)\left(\sum_j a_j X^j\right) = \sum_n \left(\sum_{i+j=n} a_i b_j\right) X^n$$

is a sum with only a finite number of nonzero terms in any case, so the product is well defined.

$U(k[[X]])$ is the set of $\sum\limits_n a_n X^n$ with $a_0 \in k^*$. For suppose $\left(\sum\limits_i a_i X^i\right)\left(\sum\limits_j a_j X^j\right) = 1$. Using the above formula yields

$$a_0 b_0 = 1, \quad \sum_{i+j=1} a_i b_j = 0 \quad \text{for}\, n > 0.$$

The first equation may be solved for b_0 if and only if a_0 is a unit in k. In that case, the remaining equations may then be solved recursively $b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n b_0)$ and the resulting formal series $\sum_n b_n X^n$ is easily seen to be the inverse of $\sum_n a_n X^n$. It follows that the complement of $U(k[[X]])$ is the set of $\sum_n a_n X^n$ with $a_n = 0$, and that is an ideal, the ideal generated by X.

Hence, $k[[X]]$ is a local ring with unique maximal ideal, the ideal generated by X. $k[[X]]$ is called the ring of formal power series in the indeterminate X. □