Answer to Question #82985, Math / Calculus

Question

Integrate $\frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}$

Solution

We have to find integral $\int \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}$

Substitute $u = \sqrt{a^2+x^2}$

$$du = \frac{2x}{2\sqrt{a^2+x^2}} = \frac{x}{\sqrt{a^2+x^2}}$$

$$\sqrt{a^2-x^2} = \sqrt{a^2-(u^2-a^2)} = \sqrt{2a^2-u^2}$$

Using this in given integral gives

$$\int \sqrt{2a^2-u^2}\, du$$

Apply Trig substitution $u = \sqrt{2}\, a\, sin\,(v)$ we get

$$\int \sqrt{2a^2-(2a^2 sin^2\, v)}\, du$$

$$\int \sqrt{2a^2(1-sin^2\, v)}\, (\sqrt{2}a\, cos\, v)\, dv$$

$$\int \sqrt{2a^2 cos^2 v}\, (\sqrt{2}a\, cos\, v) dv$$

$$\int \sqrt{2}a\, cosv\, (\sqrt{2}a\, cos\, v) dv$$

$$\int 2a^2 cos^2(v) dv$$

$$2a^2 \int cos^2(v) dv$$

$$2a^2 \int \frac{1+cos\, 2v}{2} dv$$

$$2a^2 \cdot \frac{1}{2}[\int dv + \int (cos\, 2v) dv]$$

$$a^2[v + \frac{1}{2}\ sin\ 2v]$$

Substitute back $v = sin^{-1}\left(\frac{1}{\sqrt{2}\ a}\ u\right), u = \sqrt{a^2 + x^2}$ we get

$$a^2[sin^{-1}\left(\frac{1}{\sqrt{2}\ a}\ \sqrt{a^2 + x^2}\right) + \frac{1}{2}\ sin(2sin^{-1}\left(\frac{1}{\sqrt{2}\ a}\ \sqrt{a^2 + x^2}\right)]$$

$$a^2[sin^{-1}\left(\frac{\sqrt{a^2 + x^2}}{\sqrt{2}\ a}\right) + \frac{1}{2}\ sin(2sin^{-1}\left(\frac{\sqrt{a^2 + x^2}}{\sqrt{2}\ a}\right)]$$

Hence,

$$\int \frac{x\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}} = a^2[sin^{-1}\left(\frac{\sqrt{a^2 + x^2}}{\sqrt{2}\ a}\right) + \frac{1}{2}\ sin\left(2sin^{-1}\left(\frac{\sqrt{a^2 + x^2}}{\sqrt{2}\ a}\right)\right)\Bigg] + c$$

Where c is integration constant.