Answer on Question #82973 – Math – Analytic Geometry

Question

Show that the points (-1, -2), (5,4), (-3,0) are the vertices of a right triangle, and find its area.

Solution

Let points be A(-1, -2), B(5, 4) and C(-3,0). Now let's check whether the points are on the one straight line. Find equation of the straight line AB:

$$\frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B}$$

$$\frac{x - 5}{-1 - 5} = \frac{y - 4}{-2 - 4}$$

$$\frac{x - 5}{-6} = \frac{y - 4}{-6}$$

$$x - 5 = y - 4$$

$$y = x - 1$$

If point C belongs to the line AB, its coordinates satisfies the equation: $y = x - 1$:

$$0 = -3 - 1$$

$$0 \neq -4$$

We make sure that the point C does not belong to the line AB. It also means that these points form the triangle ABC. Now let's check if it is right.

Find the coordinates of vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \{x_B - x_A, y_B - y_A\} = \{5 - (-1), 4 - (-2)\} = \{6,6\}$$

$$\overrightarrow{BC} = \{x_C - x_B, y_C - y_B\} = \{-3 - 5, 0 - 4\} = \{-8, -4\}$$

$$\overrightarrow{AC} = \{x_C - x_A, y_C - y_A\} = \{-3 - (-1), 0 - (-2)\} = \{-2,2\}$$

Now find angles between vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ (named α), $\overrightarrow{BC}$ and $\overrightarrow{AC}$ (named β), $\overrightarrow{AC}$ and $\overrightarrow{AB}$ (named γ):

$$\alpha = \arccos\left(\frac{x_{AB} * x_{BC} + y_{AB} * y_{BC}}{|AB| * |BC|}\right) = \arccos\left(\frac{6 * (-8) + 6 * (-4)}{\sqrt{6^2 + 6^2} * \sqrt{(-8)^2 + (-4)^2}}\right) =$$

$$= \arccos\left(-\frac{72}{6*\sqrt{2}*4*\sqrt{5}}\right) = \arccos(-\frac{3}{\sqrt{10}});$$

$\beta = \arccos\left(\frac{x_{AC}*x_{BC}+y_{AC}*y_{BC}}{|AC|*|BC|}\right) = \arccos\left(\frac{(-2)*(-8)+2*(-4)}{\sqrt{(-2)^2+2^2}*\sqrt{(-8)^2+(-4)^2}}\right) =$

$= \arccos\left(\frac{8}{2*\sqrt{2}*4*\sqrt{5}}\right) = \arccos(\frac{1}{\sqrt{10}});$

$\gamma = \arccos\left(\frac{x_{AB}*x_{AC}+y_{AB}*y_{AC}}{|AB|*|AC|}\right) = \arccos\left(\frac{6*(-2)+6*2}{\sqrt{6^2+6^2}*\sqrt{(-2)^2+2^2}}\right) = \arccos\left(\frac{0}{6*\sqrt{2}*2*\sqrt{2}}\right) =$

$= \arccos(0) = 90°;$

We proved that the triangle ABC is right with $\gamma = 90°$. It means that segments AB and AC are legs of the triangle ABC.

Thus, the area of the triangle ABC is

$S_{\Delta ABC} = \frac{1}{2} \cdot |AB| \cdot |AC| = \frac{1}{2} \cdot \sqrt{6^2 + 6^2} \cdot \sqrt{(-2)^2 + 2^2} = \frac{1}{2} \cdot 6\sqrt{2} \cdot 2\sqrt{2} =$

$= 6 \cdot 2 = 12\ square\ units.$

Answer: the points (-1, -2), (5,4), (-3,0) are the vertices of a right triangle, and its area is 12 square units.