Question #8297 Let X be complex Banach space , $T \in B(X, X)$ and p a polynomial .Show that the equation $p(T)x = y$ has a unique solution x for every $y \in X$ if and only if $p(\lambda) \neq 0$, for all $\lambda \in \sigma(T)$.

Solution. If $p(\lambda) \neq 0$ for $\lambda in \sigma(T)$ then the function $g = 1/p$ is holomorphic on $\sigma(T)$. Obviously $pg = 1$, and by the Spectral Theorem p(T)g(T) = I, and so $p(T)$ is invertible, which entails the statement of the problem. On the other hand, if there is a unique solution then $p(T)$ is invertible. Assume that $p(\lambda) = 0$ for some $\lambda \in \sigma(T)$, then it give us $p(z) = (z - \lambda)f(z)$, which gives $(T - I)f(T) = f(T) = f(T)(T - I)$: But since $T - I$ would not be invertible, we would have that p(T) is not invertible, and this is a contradiction and so $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.