Answer on Question #82880 – Math – Linear Algebra

Question

Solve the following equations using matrix algebra:

$$\begin{aligned} 2x + y - z &= 11 \\ x - 2y + 2z &= -2 \\ 3x - y + 3z &= 5 \end{aligned}$$

Solution

Method 1

Write the linear system in matrix form $AX = B$, where A is coefficient matrix of the linear system:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & 3 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix}.$$

Calculate the determinant of the coefficient matrix A :

$$det(A) = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & 3 \end{vmatrix} = 2 \cdot (-2) \cdot 3 + 1 \cdot 3 \cdot 2 + 1 \cdot (-1) \cdot (-1) - (-1) \cdot (-2) \cdot 3 -$$

$$-1 \cdot 1 \cdot 3 - 2 \cdot 2 \cdot (-1) = -12 + 6 + 1 - 6 - 3 + 4 = -10 \neq 0.$$

So linear system $AX = B$ has only one solution, because $det(A) \neq 0, det(A) = D = -10$.

We can find the X matrix by multiplying the inverse of the A matrix by the B matrix:

$$X = A^{-1} B.$$

First, we need to find the inverse of the A matrix. The A matrix is invertible because $det(A) = -10 \neq 0$. Use the adjoint method to find A^{-1}. Calculate the cofactors of each element.

$$c_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 2 \\ -1 & 3 \end{vmatrix} = -6 + 2 = -4, c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = -1(3 - 6) = 3,$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 + 6 = 5, \quad c_{21} = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = -1(3-1) = -2,$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} = 6 + 3 = 9, \quad c_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -1(-2-3) = 5,$$

$$c_{31} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} = 2 - 2 = 0, c_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -1(4+1) = -5,$$

$$c_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5.$$

Write cofactor matrix C:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{21} & {}_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \text{ so } C = \begin{bmatrix} -4 & 3 & 5 \\ -2 & 9 & 5 \\ 0 & -5 & -5 \end{bmatrix} \; adj\, A = C^T = \begin{bmatrix} -4 & -2 & 0 \\ 3 & 9 & -5 \\ 5 & 5 & -5 \end{bmatrix},$$

$$A^{-1} = \frac{1}{\det(B)} adj\, A, \; A^{-1} = -\frac{1}{10} \begin{bmatrix} -4 & -2 & 0 \\ 3 & 9 & -5 \\ 5 & 5 & -5 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{3}{10} & -\frac{9}{10} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then find $X = A^{-1}B$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -4 & -2 & 0 \\ 3 & 9 & -5 \\ 5 & 5 & -5 \end{bmatrix} \begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -4\cdot 11 - 2\cdot(-2) + 0\cdot 5 \\ 3\cdot 11 - 9\cdot 2 - 5\cdot 5 \\ 5\cdot 11 - 5\cdot 2 - 5\cdot 5 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -40 \\ -10 \\ 20 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

Answer:

$x = 4, y = 1, z = -2.$

Method 2

We also can use Cramer's Rule to solve the linear system, where

$$x = \frac{D_x}{D}, = \frac{D_y}{D}, z = \frac{D_z}{D},$$

and the determinants D_x, D_y, D_z are determined by replacing the coefficient of x or y or z in the matrix A by the constant terms B.

$D = |A| = -10.$ Find D_x, D_y, D_z:

$$D_x = \begin{vmatrix} 11 & 1 & -1 \\ -2 & -2 & 2 \\ 5 & -1 & 3 \end{vmatrix} = 11 \cdot (-2) \cdot 3 + 1 \cdot 5 \cdot 2 - 2 \cdot (-1) \cdot (-1) + 1 \cdot (-2) \cdot 5 -$$

$$-1 \cdot 3 \cdot (-2) - 11 \cdot (-1) \cdot 2 = -66 + 10 - 2 - 10 + 6 + 22 = -40,$$

$$D_y = \begin{vmatrix} 2 & 11 & -1 \\ 1 & -2 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 2 \cdot (-2) \cdot 3 + 11 \cdot 3 \cdot 2 - 1 \cdot 1 \cdot 5 + 1 \cdot (-2) \cdot 3 - 2 \cdot 5 \cdot 2 - 3 \cdot 11 \cdot 1 =$$

$$= -12 + 66 - 5 - 6 - 20 - 33 = -10,$$

$$D_z = \begin{vmatrix} 2 & 1 & 11 \\ 1 & -2 & -2 \\ 3 & -1 & 5 \end{vmatrix} = 2 \cdot (-2) \cdot 5 + 1 \cdot 11 \cdot (-1) + 1 \cdot 3 \cdot (-2) - 11 \cdot (-2) \cdot 3 - 1 \cdot 5 \cdot 1 -$$

$$-2 \cdot (-1) \cdot (-2) = -20 - 11 - 6 + 66 - 5 - 4 = 20.$$

$$x = \frac{D_x}{D} = \frac{-40}{-10} = 4,$$

$$y = \frac{D_y}{D} = \frac{-10}{-10} = 1,$$

$$z = \frac{D_y}{D} = \frac{20}{-10} = -2.$$

Answer:

$x = 4, y = 1, z = -2.$