Answer on Question #82740 – Math – Abstract Algebra

Question

R be a commutative ring with identity iff R[[X]] is commutative ring with identity.

Solution

1. Sufficiency.

Suppose $R[[x]]$ is commutative. Let $f: R \to R[[x]]$ is $f(r) = r + 0x + 0x^2 + 0x^3 + \cdots$. It is the ring homomorphism, since $f(r+s) = f(r) + f(x)$ and $f(rs) = f(r)f(s)$. Moreover, $f(x)$ is surjection, since if $r \neq s$ then $f(r) \neq f(s)$. So, R is isomorphic to the subring of constant terms of $R[[x]]$, and since $R[[x]]$ is commutative, then R is commutative too.

2. Necessity.

Suppose R is commutative. Let $f(x), g(x) \in R[[x]]$, $a(x) = f(x)g(x)$ and $b(x) = g(x)f(x)$. Then, since f_n and g_n belong to the commutative R:

$$a_n = \sum_{i=0}^{n} f_i g_{n-i} = \sum_{i=0}^{n} g_i f_{n-i} = b_n$$

So, all coefficients of $a(x)$ and $b(x)$ are equal, and then:

$$f(x)g(x) = a(x) = b(x) = g(x)f(x)$$