Answer on Question # 82739, Math / Combinatorics | Number Theory

Question 1. $A = \{1, 2, 3, ..., 2016, 2017, 2018\}$, *$S$ is a set whose elements are the subsets of A such that one element of S cannot be a subset of another element. Let, S has maximum possible number of elements. In this case, what is the number of elements of S?*

Solution. Consider the general case: $|A| = 2n$. Say that two subsets of A are *incomparable* if neither is a subset of the other, and say that a subset of A is *large* if it has more than n elements. Let $\mathcal{A}$ be any pairwise incomparable family of subsets of A. For any set X let X_n be the family of subsets of X of cardinality n. Let

$$\mathcal{B} = \{U \in \mathcal{A} \colon |U| \leqslant n\} \cup \bigcup \{U_n \colon U \in \mathcal{A},\ |U| > n\}.$$

$\mathcal{B}$ is simply the result of replacing each large member of $\mathcal{A}$ by its n-element subsets. $\mathcal{B}$ is pairwise incomparable, and clearly $|\mathcal{B}| \geqslant |\mathcal{A}|$. The strategy is easy: replace big sets with their n-element subsets, do an inverse, replace big sets again.

Now let $\mathcal{C} = \{A \setminus B \colon B \in \mathcal{B}\}$. $\mathcal{C}$ is pairwise incomparable, $|\mathcal{C}| = |\mathcal{B}|$, and $|C| \geqslant n$ for each $C \in \mathcal{C}$.

Repeat the process used to go from $\mathcal{A}$ to $\mathcal{B}$. Let

$$\mathcal{D} = (\mathcal{C} \cap A_n) \cup \bigcup \{C_n \colon C \in \mathcal{C} \setminus A_n\}.$$

Then $|\mathcal{D}| \geqslant |\mathcal{C}| \geqslant |\mathcal{A}|$, and $\mathcal{D} \subset A_n$, so $|\mathcal{A}| \leqslant |A_n| = \binom{2n}{n}$, so $\binom{2n}{n}$ is indeed an upper bound on the size of any family of pairwise incomparable subsets of A. Since A_n is a pairwise incomparable family of cardinality $\binom{2n}{n}$, this upper bound is sharp.

Coming back to our case: $n = 1009$, $S = A_{1009}$, $|S| = \binom{2018}{1009}$. □