Answer on Question #82698 – Math – Abstract Algebra

Question

Let R be a commutative ring with identity. A formal power series $f(X)$ is invertible in $R[[x]]$ if and only if the constant term f_0 has an inverse in R.

Solution

1. Necessity.

If $f(X)$ is invertible, then exists $g(X) \in R[[x]]$, such that $f(X)g(X) = 1$, and the constant term of $f(X)g(X)$ is equal to $f_0 g_0$, so $f_0 g_0 = 1$, so f_0 is invertible.

2. Sufficiency.

Suppose f_0 is invertible, so $\frac{1}{f_0}$ exists in R.

Let's define g_n recursively, by induction: $g_0 \coloneqq \frac{1}{f_0}, g_n \coloneqq -\frac{1}{f_0}(\sum_{i=1}^{n} f_i g_{n-i})$. It can be defined because $(n-i) < n$, so all g_{n-i} are already defined.

We will prove that $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$ is the inverse of $f(x)$, so $f(x)g(x) = 1$

Let h_i be the coefficient of the n-th power of the x in the $f(x)g(x)$. Let's prove that

$h_0 = 1$ and $h_i = 0$ for all $i \in \boldsymbol{N}$:

$$h_0 = f_0 g_0 = f_0 \cdot \frac{1}{f_0} = 1$$

$$h_n = \sum_{i=0}^{n} f_i g_{n-i} = f_0 g_n + \sum_{k=1}^{n} f_i g_{n-i} = f_0 \cdot (-\frac{1}{f_0})\left(\sum_{i=1}^{n} f_i g_{n-i}\right) + \sum_{i=1}^{n} f_i g_{n-i} = 0$$

So, $f(x)g(x) = 1$.