Answer on Question # 82678 - Math - Real Analysis

Question 1. *a) Is the sequence a Cauchy sequence or not?*
b) $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $x_1 < x_2$. *Show that* $\{x_n\}_{n\in\mathbb{N}}$ *converges and find the limit.*

Solution. a) We can use the statement: *a sequence converges if and only if it is a Cauchy sequence.* Proof of this statement: `http://www.math.stonybrook.edu/~mde/319S_08/testsolII.pdf`

(i) $\left\{n + \frac{1}{n} \colon n \in \mathbb{N}\right\}$ diverges because $n + \frac{1}{n} \to \infty, n \to \infty$. So it is not a Cauchy sequence.

(ii) $\left\{1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \colon n \in \mathbb{N}\right\}$ converges because we have the Taylor series $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ and for $x = 1$ we have $1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \to e - 1, n \to \infty$. So it is a Cauchy sequence.

(iii) $\{(-1)^n \colon n \in \mathbb{N}\}$ diverges because $(-1)^{2k} \to 1$ and $(-1)^{2k-1} \to -1, k \to \infty$. So it is not a Cauchy sequence.

(iv) $\left\{n + \frac{(-1)^n}{n} \colon n \in \mathbb{N}\right\}$ diverges because $n + \frac{(-1)^n}{n} \to \infty, n \to \infty$. So it is not a Cauchy sequence.

b) You can prove convergence using the Cauchy criterion - the distance between consecutive terms halves every term: $|x_n - x_{n-1}| = \frac{1}{2}|x_{n-1} - x_{n-2}|$, so iterating this equality to obtain $|x_n - x_{n-1}| = 2^{-n+2}|x_2 - x_1|$ we can use a geometric series bound $\left(\text{i.e. use } \sum\limits_{n=0}^{\infty} r^n = \dfrac{1}{1-r} \text{ for } |r| < 1\right)$ to show the sequence is Cauchy:

$$|x_{n+m} - x_n| \leqslant \sum_{k=1}^{m} |x_{n+k} - x_{n+k-1}| \leqslant 2^{-n}|x_2 - x_1| \sum_{k=1}^{m} 2^{-k} \leqslant 2^{-n}|x_2 - x_1| \to 0,\ n \to \infty.$$

To find the limit, write the sequence as a geometric series similar to the above:

$$\lim_{n\to\infty} x_n = x_1 + \sum_{n=2}^{\infty}(x_n - x_{n-1}) = x_1 + (x_2 - x_1)\sum_{n=2}^{\infty}(-2)^{-n+2} = x_1 + (x_2 - x_1)\frac{1}{1-(-2)} = \frac{x_1}{3} + \frac{2x_2}{3}.$$

□