Answer on Question #82677 – Math – Real Analysis

Question

Establish the convergence or divergence of the sequence (Xn) such that
I) Xn= 1/1^2+1/2^2+1/3^2+.......1/n^2 for all n belongs to N.
ii) Xn= (1+1/n)^n+1

iii) Xn= (1+1/n+1)^n
Iv) Xn= (1-1/n)^n

Solution

1) $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$

It is obvious that this sequence is monotonous. Let us prove that it is bounded.

$$x_n < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n} = 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}$$
$$= 2 - \frac{1}{n} < 2$$

Sequence is monotonous and bounded, and therefore converges.

2a) $x_n = \left(1 + \frac{1}{n}\right)^n + 1,$

$\lim\limits_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n + 1 = e + 1.$ Hence, it converges.

2b) $x_n = \left(1 + \frac{1}{n}\right)^{n+1},$

$\lim\limits_{n\to+\infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim\limits_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) = e \cdot 1 = e$, hence it converges.

3a) $x_n = \left(1 + \frac{1}{n} + 1\right)^n$

$$x_n = \left(2 + \frac{1}{n}\right)^n > 2^n$$

x_n does not converge, or, in other words, converges to $+\infty$.

3b) $x_n = \left(1 + \frac{1}{n+1}\right)^n$

$$\lim_{n\to+\infty} \left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to+\infty} \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} = \frac{e}{1} = e$$

It converges.

4) $x_n = \left(1 - \frac{1}{n}\right)^n$

$$\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n-1+1}\right)^n = \frac{1}{\left(1+\frac{1}{n-1}\right)^n} = \frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right)},$$

hence

$$\lim_{n\to+\infty} x_n = \frac{1}{e\cdot 1} = \frac{1}{e}.$$

Thus, x_n converges.