Answer to Question #82609, Math / Abstract Algebra

Question. Prove that $\mathbb{R}^n/\mathbb{R}^m \cong \mathbb{R}^{n-m}$ as groups, where n, m belongs to $\mathbb{N}$ such that $n \geq m$.

Answer. Define a projection group homomorphism $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ by

$$f((a_1, \dots, a_m, a_{m+1}, \dots, a_n)) = (a_{m+1}, \dots, a_n).$$

We will use the First Group Isomorphism Theorem.

- The kernel of f is the set of all elements a of $\mathbb{R}^n$ having the form $(a_1, \dots, a_n)$ such that all of the equivalent conditions below hold:
  - $f((a_1, \dots, a_m, a_{m+1}, \dots, a_n)) = (0, \dots, 0)$;
  - $(a_{m+1}, \dots, a_n) = (0, \dots, 0)$;
  - for all $i \in \{m+1, \dots, n\}$, $a_i = 0$;
  - a has the form $(a_1, \dots, a_m, 0, \dots, 0)$.

  Clearly, $\ker f$ has dimension m and is $\mathbb{R}^m$ viewed as a subgroup of $\mathbb{R}^n$.

- For every $(a_{m+1}, \dots, a_n) \in \mathbb{R}^{n-m}$,

  $$f((0, \dots, 0, a_{m+1}, \dots, a_n)) = (a_{m+1}, \dots, a_n).$$

  Hence f is surjective, and the range (image) of f is $\mathbb{R}^{n-m}$.

By the First Group Isomorphism Theorem, $\mathbb{R}^n/\mathbb{R}^m \cong \mathbb{R}^{n-m}$.