Answer to Question #82540 - Math / Abstract Algebra

Question. Prove that an element of an integral domain is a unit iff it generates the domain.

Answer. By "r generates I" here we mean that r generates an ideal I, in other words, I is the least ideal containing r.

Let R be an integral domain, and $a \in R$.

- ($\Longrightarrow$) Assume that a is a unit. Let I be an ideal of R containing a. Let $b \in R$. As $b = (ba^{-1})a$ and $a \in I$, by the properties of ideal, $b \in I$. Hence $I = R$. As I was arbitrary, every ideal containing a is equal to R. Hence R is the least ideal containing a, in other words, a generates R.

- ($\Longleftarrow$) Assume that a generates R. The set $I = \{ra \mid r \in R\}$ is an ideal of R as shown below.

  - If $ra, sa \in I$ for some $r, s \in R$, then $ra + sa = (r + s)a \in I$.
  - $0 = 0 \cdot a \in I$.
  - If $ra \in I$ for some $r \in R$, then $-(ra) = (-r)a \in I$.
  - If $ra \in I$ for some $r \in R$, and $s \in R$, then $s(ra) = (sr)a \in I$.

  Also $a = 1 \cdot a \in I$. Hence I is an ideal containing a. As R is the least ideal containing a, I includes R, so I contains 1. Hence there is $r \in R$ such that $1 = ra$. In other words, a is a unit.