Answer on Question #82533 – Math – Linear Algebra

Question

a. Find a basis for the span of the following set of vectors

|1 | |-4 | |1 | |5| |2| | -1| |11|

|3 | |-12| |-4| |8| |11| | -7| | 52 |

|-1| | 4 | |-5| |-17| |-21| |19| | -80|

|2| |-8| |4| |16| |17| | -3| | 82|

Solution

a

$$\begin{pmatrix} 1 & -4 & 1 & 5 & 2 & -1 & 11 \\ 3 & -12 & -4 & 8 & 11 & -7 & 52 \\ -1 & 4 & -5 & -17 & -21 & 19 & -80 \\ 2 & -8 & 4 & 16 & 17 & -3 & 82 \end{pmatrix} \to \begin{matrix} r2 = r2 + (-3)r1 \\ r3 = r3 + r1 \\ r4 = r4 + (-2)r1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -4 & 1 & 5 & 2 & -1 & 11 \\ 0 & 0 & -7 & -7 & 5 & -4 & 19 \\ 0 & 0 & -4 & -12 & -19 & 18 & -69 \\ 0 & 0 & 2 & 6 & 13 & -1 & 60 \end{pmatrix} \to \begin{matrix} r1 = r1 + \left(\frac{1}{7}\right)r2 \\ r3 = r3 + \left(\frac{-4}{7}\right)r2 \\ r4 = r4 + \left(\frac{2}{7}\right)r2 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -4 & 0 & 4 & \frac{19}{7} & -\frac{11}{7} & \frac{96}{7} \\ 0 & 0 & -7 & -7 & 5 & -4 & 19 \\ 0 & 0 & 0 & -8 & -\frac{153}{7} & \frac{142}{7} & -\frac{559}{7} \\ 0 & 0 & 0 & 4 & \frac{101}{7} & -\frac{15}{7} & \frac{458}{7} \end{pmatrix} \to \begin{matrix} r1 = r1 + \left(\frac{1}{2}\right)r3 \\ r2 = r2 + \left(\frac{-7}{8}\right)r3 \\ r4 = r4 + \left(\frac{1}{2}\right)r3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -4 & 0 & 0 & -\frac{115}{14} & \frac{60}{7} & -\frac{367}{14} \\ 0 & 0 & -7 & 0 & \frac{193}{8} & -\frac{87}{4} & \frac{711}{8} \\ 0 & 0 & 0 & -8 & -\frac{153}{7} & \frac{142}{7} & -\frac{559}{7} \\ 0 & 0 & 0 & 0 & \frac{7}{2} & 8 & \frac{51}{2} \end{pmatrix} \to \begin{matrix} r1 = r1 + \left(\frac{115}{49}\right)r4 \\ r2 = r2 + \left(\frac{-193}{28}\right)r4 \\ r3 = r3 + \left(\frac{306}{49}\right)r4 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & \frac{1340}{49} & \frac{1648}{49} \\ 0 & 0 & -7 & 0 & 0 & -\frac{2153}{28} & -\frac{2433}{28} \\ 0 & 0 & 0 & -8 & 0 & \frac{3442}{49} & \frac{3890}{49} \\ 0 & 0 & 0 & 0 & \frac{7}{2} & 8 & \frac{51}{2} \end{pmatrix}$$

The first, the third, the fourth, the fifth columns are linearly independent, hence a basis is:

$$B = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ -17 \\ 16 \end{pmatrix}, \begin{pmatrix} 2 \\ 11 \\ -21 \\ 17 \end{pmatrix} \right\}$$

Answer a.: basis $\left\{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix},\begin{pmatrix}1\\-4\\-5\\4\end{pmatrix},\begin{pmatrix}5\\8\\-17\\16\end{pmatrix},\begin{pmatrix}2\\11\\-21\\17\end{pmatrix}\right\}$.

Question

b. find the coordinate vector [x]B for the vector (see below) using the basis from a above

|-8|

|-58|

|151|

|-58|

Solution

b

$$x = P_B\,[x]_B \;\rightarrow\; [x]_B = P_B^{-1}\,x$$

$$x = \begin{pmatrix}-8\\-58\\151\\-58\end{pmatrix}, B = \left\{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix},\begin{pmatrix}1\\-4\\-5\\4\end{pmatrix},\begin{pmatrix}5\\8\\-17\\16\end{pmatrix},\begin{pmatrix}2\\11\\-21\\17\end{pmatrix}\right\} \rightarrow P_B = \begin{pmatrix}1 & 1 & 5 & 2\\3 & -4 & 8 & 11\\-1 & -5 & -17 & -21\\2 & 4 & 16 & 17\end{pmatrix}$$

$$\det P_B = \begin{vmatrix}1 & 1 & 5 & 2\\3 & -4 & 8 & 11\\-1 & -5 & -17 & -21\\2 & 4 & 16 & 17\end{vmatrix} = \begin{Bmatrix}r2 = r2 + (-3)r1\\r3 = r3 + r1\\r4 = r4 + (-2)r1\end{Bmatrix} = \begin{vmatrix}1 & 1 & 5 & 2\\0 & -7 & -7 & 5\\0 & -4 & -12 & -19\\0 & 2 & 6 & 13\end{vmatrix} = \begin{Bmatrix}r3 = r3 + \left(\frac{-4}{7}\right)r2\\r4 = r4 + \left(\frac{2}{7}\right)r2\end{Bmatrix}$$

$$= \begin{vmatrix}1 & 1 & 5 & 2\\0 & -7 & -7 & 5\\0 & 0 & -8 & -153/7\\0 & 0 & 4 & 101/7\end{vmatrix} = \left\{r4 = r4 + \left(\frac{1}{2}\right)r3\right\} = \begin{vmatrix}1 & 1 & 5 & 2\\0 & -7 & -7 & 5\\0 & 0 & -8 & -153/7\\0 & 0 & 4 & 7/2\end{vmatrix}$$

$$= \frac{1\cdot(-7)\cdot(-8)\cdot 7}{2} = 196$$

$$P_{11} = \begin{vmatrix}-4 & 8 & 11\\-5 & -17 & -21\\4 & 16 & 17\end{vmatrix}$$

$$= (-4)\cdot(-17)\cdot 17 + 8\cdot(-21)\cdot 4 + 11\cdot(-5)\cdot 16 - 11\cdot(-17)\cdot 4 - (-4)\cdot(-21)\cdot 16 - 8\cdot(-5)\cdot 17 = 1156 - 672 - 880 + 748 - 1344 + 680 = -312$$

$$P_{12} = \begin{vmatrix} 3 & 8 & 11 \\ -1 & -17 & -21 \\ 2 & 16 & 17 \end{vmatrix}$$

$$= 3 \cdot (-17) \cdot 17 + 8 \cdot (-21) \cdot 2 + 11 \cdot (-1) \cdot 16 - 11 \cdot (-17) \cdot 2 - 3 \cdot (-21) \cdot 16 - 8 \cdot (-1) \cdot 17 = -867 - 336 - 176 + 374 + 1008 + 136 = 139$$

$$P_{13} = \begin{vmatrix} 3 & -4 & 11 \\ -1 & -5 & -21 \\ 2 & 4 & 17 \end{vmatrix}$$

$$= 3 \cdot (-5) \cdot 17 + (-4) \cdot (-21) \cdot 2 + 11 \cdot (-1) \cdot 4 - 11 \cdot (-5) \cdot 2 - 3 \cdot (-21) \cdot 4 - (-4) \cdot (-1) \cdot 17 = -255 + 168 - 44 + 110 + 252 - 68 = 163$$

$$P_{14} = \begin{vmatrix} 3 & -4 & 8 \\ -1 & -5 & -17 \\ 2 & 4 & 16 \end{vmatrix}$$

$$= 3 \cdot (-5) \cdot 16 + (-4) \cdot (-17) \cdot 2 + 8 \cdot (-1) \cdot 4 - 8 \cdot (-5) \cdot 2 - 3 \cdot (-17) \cdot 4 - (-4) \cdot (-1) \cdot 16 = -240 + 136 - 32 + 80 + 204 - 64 = 84$$

$$P_{21} = \begin{vmatrix} 1 & 5 & 2 \\ -5 & -17 & -21 \\ 4 & 16 & 17 \end{vmatrix}$$

$$= 1 \cdot (-17) \cdot 17 + 5 \cdot (-21) \cdot 4 + 2 \cdot (-5) \cdot 16 - 2 \cdot (-17) \cdot 4 - 1 \cdot (-21) \cdot 16 - 5 \cdot (-5) \cdot 17 = -289 - 420 - 160 + 136 + 336 + 425 = 28$$

$$P_{22} = \begin{vmatrix} 1 & 5 & 2 \\ -1 & -17 & -21 \\ 2 & 16 & 17 \end{vmatrix}$$

$$= 1 \cdot (-17) \cdot 17 + 5 \cdot (-21) \cdot 2 + 2 \cdot (-1) \cdot 16 - 2 \cdot (-17) \cdot 2 - 1 \cdot (-21) \cdot 16 - 5 \cdot (-1) \cdot 17 = -289 - 210 - 32 + 68 + 336 + 85 = -42$$

$$P_{23} = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -5 & -21 \\ 2 & 4 & 17 \end{vmatrix}$$

$$= 1 \cdot (-5) \cdot 17 + 1 \cdot (-21) \cdot 2 + 2 \cdot (-1) \cdot 4 - 2 \cdot (-5) \cdot 2 - 1 \cdot (-21) \cdot 4 - 1 \cdot (-1) \cdot 17 = -85 - 42 - 8 + 20 + 84 + 17 = -14$$

$$P_{24} = \begin{vmatrix} 1 & 1 & 5 \\ -1 & -5 & -17 \\ 2 & 4 & 16 \end{vmatrix}$$

$$= 1 \cdot (-5) \cdot 16 + 1 \cdot (-17) \cdot 2 + 5 \cdot (-1) \cdot 4 - 5 \cdot (-5) \cdot 2 - 1 \cdot (-17) \cdot 4 - 1 \cdot (-1) \cdot 16 = -80 - 34 - 20 + 50 + 68 + 16 = 0$$

$$P_{31} = \begin{vmatrix} 1 & 5 & 2 \\ -4 & 8 & 11 \\ 4 & 16 & 17 \end{vmatrix}$$

$$= 1 \cdot 8 \cdot 17 + 5 \cdot 11 \cdot 4 + 2 \cdot (-4) \cdot 16 - 2 \cdot 8 \cdot 4 - 1 \cdot 11 \cdot 16 - 5 \cdot (-4) \cdot 17$$

$$= 136 + 220 - 128 - 64 - 176 + 340 = 328$$

$$P_{32}=\begin{vmatrix}1&5&2\\3&8&11\\2&16&17\end{vmatrix}=1\cdot8\cdot17+5\cdot11\cdot2+2\cdot3\cdot16-2\cdot8\cdot2-1\cdot11\cdot16-5\cdot3\cdot17$$
$$=136+110+96-32-176-255=-121$$

$$P_{33}=\begin{vmatrix}1&1&2\\3&-4&11\\2&4&17\end{vmatrix}=1\cdot(-4)\cdot17+1\cdot11\cdot2+2\cdot3\cdot4-2\cdot(-4)\cdot2-1\cdot11\cdot4-1\cdot3\cdot17$$
$$=-68+22+24+16-44-51=-101$$

$$P_{34}=\begin{vmatrix}1&1&5\\3&-4&8\\2&4&16\end{vmatrix}=1\cdot(-4)\cdot16+1\cdot8\cdot2+5\cdot3\cdot4-5\cdot(-4)\cdot2-1\cdot8\cdot4-1\cdot3\cdot16$$
$$=-64+16+60+40-32-48=-28$$

$$P_{41}=\begin{vmatrix}1&5&2\\-4&8&11\\-5&-17&-21\end{vmatrix}$$
$$=1\cdot8\cdot(-21)+5\cdot11\cdot(-5)+2\cdot(-4)\cdot(-17)-2\cdot8\cdot(-5)-1\cdot11\cdot(-17)$$
$$-5\cdot(-4)\cdot(-21)=-168-275+136+80+187-420=-460$$

$$P_{42}=\begin{vmatrix}1&5&2\\3&8&11\\-1&-17&-21\end{vmatrix}$$
$$=1\cdot8\cdot(-21)+5\cdot11\cdot(-1)+2\cdot3\cdot(-17)-2\cdot8\cdot(-1)-1\cdot11\cdot(-17)-5$$
$$\cdot3\cdot(-21)=-168-55-102+16+187+315=193$$

$$P_{43}=\begin{vmatrix}1&1&2\\3&-4&11\\-1&-5&-21\end{vmatrix}$$
$$=1\cdot(-4)\cdot(-21)+1\cdot11\cdot(-1)+2\cdot3\cdot(-5)-2\cdot(-4)\cdot(-1)-1\cdot11\cdot(-5)$$
$$-1\cdot3\cdot(-21)=84-11-30-8+55+63=153$$

$$P_{44}=\begin{vmatrix}1&1&5\\3&-4&8\\-1&-5&-17\end{vmatrix}$$
$$=1\cdot(-4)\cdot(-17)+1\cdot8\cdot(-1)+5\cdot3\cdot(-5)-5\cdot(-4)\cdot(-1)-1\cdot8\cdot(-5)$$
$$-1\cdot3\cdot(-17)=68-8-75-20+40+51=56$$

$$P^*=\begin{pmatrix}P_{11}&-P_{12}&P_{13}&-P_{14}\\-P_{21}&P_{22}&-P_{23}&P_{24}\\P_{31}&-P_{32}&P_{33}&-P_{34}\\-P_{41}&P_{42}&-P_{43}&P_{44}\end{pmatrix}=\begin{pmatrix}-312&-139&163&-84\\-28&-42&14&0\\328&121&-101&28\\460&193&-153&56\end{pmatrix}$$

$$P^{*T}=\begin{pmatrix}-312&-28&328&460\\-139&-42&121&193\\163&14&-101&-153\\-84&0&28&56\end{pmatrix}$$

$$P_B^{-1} = \frac{P^{*T}}{detP_B} = \begin{pmatrix} -78/49 & -1/7 & 82/49 & 115/49 \\ -139/196 & -3/14 & 121/196 & 193/196 \\ 163/196 & 1/14 & -101/196 & -153/196 \\ -3/7 & 0 & 1/7 & 2/7 \end{pmatrix}$$

$$[x]_B = P_B^{-1}\,x = -8\begin{pmatrix} -\frac{78}{49} \\ -\frac{139}{196} \\ \frac{163}{196} \\ -\frac{3}{7} \end{pmatrix} - 58\begin{pmatrix} -\frac{1}{7} \\ -\frac{3}{14} \\ \frac{1}{14} \\ 0 \end{pmatrix} + 151\begin{pmatrix} \frac{82}{49} \\ \frac{121}{196} \\ -\frac{101}{196} \\ \frac{1}{7} \end{pmatrix} - 58\begin{pmatrix} \frac{115}{49} \\ \frac{193}{196} \\ -\frac{153}{196} \\ \frac{2}{7} \end{pmatrix} = \begin{pmatrix} \frac{6742}{49} \\ \frac{10625}{196} \\ -\frac{8493}{196} \\ \frac{59}{7} \end{pmatrix}.$$

Answer: $[x]_B = \begin{pmatrix} \frac{6742}{49} \\ \frac{10625}{196} \\ -\frac{8493}{196} \\ \frac{59}{7} \end{pmatrix}.$