Answer to Question #82485, Math / Real Analysis

Question

(1) Use definition of limit of a sequence to establish the following limits

(A) $\lim \frac{n}{n^2+1} = 0$

(B) $\lim \frac{2n}{n+1} = 2$

(C) $\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$

(D) Show that

(i) $\lim \frac{1}{3^n} = 0$ (ii) $\lim \frac{2^n}{n!} = 0$ (iii) $\lim (2n)^{\frac{1}{n}} = 1$

Solution

(A) Given $\frac{n}{n^2+1}$

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}$$

Let $\epsilon > 0$ be any number and we wish to choose N so that

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon \quad \text{whenever } n > N$$

Thus, the inequality holds if we choose $N > \frac{1}{\epsilon}$

(B) Let $\epsilon > 0$ be any number and we wish to choose N so that

$$\left|\frac{2n}{n+1} - \frac{2(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| < \frac{1}{n} < \epsilon \quad \text{whenever } n > N$$

Thus, the inequality holds if we choose $N > \frac{1}{\epsilon}$

(C) Let $\epsilon > 0$ be any number and we wish to choose N so that

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \left|\frac{3n+1}{2n} - \frac{3n}{2n}\right| = \frac{1}{2n} < \epsilon \quad \text{whenever } n > N$$

Thus, the inequality holds if we choose $N > \frac{1}{2\epsilon}$

(D)

(i) Consider the result holds for n.Then

$$\frac{1}{3^n} < 1$$

Let $\epsilon > 0$ be any number and we wish to choose N so that

$$\left|\frac{1}{3^n}\right| < 1 < \epsilon \quad \text{whenever } n > N$$

Thus, the inequality holds if we choose $N = 1$

(ii) $\frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$ if $n \geq 3$.

If $n = 3$ then $\frac{8}{6} \leq 2(\frac{2}{3})$

Consider the result holds for n.Then

$$\frac{2(2^n)}{(n+1)n!} \leq \frac{2}{n+1} \cdot (2)\left(\frac{2}{3}\right)^{n-2} \leq (2)\left(\frac{2}{3}\right)^{n-1}$$

Let $\epsilon > 0$ be any number and we wish to choose N so that

$$\left|\frac{2^n}{n!}\right| < \left|\frac{4}{n}\right| < \frac{4}{\left[\frac{4}{\epsilon}\right]+1} < \epsilon \quad \text{whenever } n > N$$

Thus, the inequality holds if we choose $N = \left[\frac{4}{\epsilon}\right] + 1$

(iii)

$2 \leq 2n^{\frac{1}{n}} \leq 2n$

$n^{1/n}$converges to 1, so the limit $\lim(2n)^{\frac{1}{n}}$converges to 1.

Hence,

$\lim(2n)^{\frac{1}{n}} = 1$