Answer on Question #82482 – Math – Differential Equations

Question

Laplace transform to solve the following ode

$4y''+y'-3y=2e^{-5t}$ initial conditions $y'(0)=y(0)=1$.

Solution

Let Y(s) be the Laplace transform of y(t). Taking the Laplace transform of the differential equation we have:

$$L[4y''+y'-3y]=L[2e^{-5t}].$$

Now

$$L[4y''+y'-3y]=4L[y'']+L[y']-3L[y]=4\cdot\left(s^2\cdot Y(s)-s\cdot y(0)-y'(0)\right)+$$
$$+s\cdot Y(s)-y(0)-3\cdot Y(s)=4\cdot s^2\cdot Y(s)-4\cdot s\cdot 1-4\cdot 1+s\cdot Y(s)-1-3\cdot Y(s)=$$
$$=Y(s)\cdot\left(4s^2+s-3\right)-4s-5;$$

$$L[2e^{-5t}]=2\cdot\frac{1}{s+5}=\frac{2}{s+5}.$$

Then

$$Y(s)\cdot\left(4s^2+s-3\right)-4s-5=\frac{2}{s+5};$$
$$Y(s)\cdot\left(4s^2+s-3\right)=\frac{2}{s+5}+4s+5;$$
$$Y(s)\cdot\left(4s^2+s-3\right)=\frac{2+\left(4s+5\right)\cdot\left(s+5\right)}{s+5};$$
$$Y(s)=\frac{2+4s^2+20s+5s+25}{\left(s+5\right)\cdot\left(4s^2+s-3\right)};$$
$$Y(s)=\frac{4s^2+25s+27}{\left(s+5\right)\cdot\left(4s^2+s-3\right)};$$
$$Y(s)=\frac{4s^2+25s+27}{\left(s+5\right)\cdot\left(s+1\right)\cdot\left(4s-3\right)}.$$

We can simplify this expression using the method of partial fractions.

$$\frac{4s^2+25s+27}{(s+5)\cdot(s+1)\cdot(4s-3)}=\frac{A}{s+5}+\frac{B}{s+1}+\frac{C}{4s-3}=$$

$$=\frac{A\cdot(s+1)\cdot(4s-3)+B\cdot(s+5)\cdot(4s-3)+C\cdot(s+1)\cdot(s+5)}{(s+5)\cdot(s+1)\cdot(4s-3)}=$$

$$=\frac{A\cdot\left(4s^2-3s+4s-3\right)+B\cdot\left(4s^2-3s+20s-15\right)+C\cdot\left(s^2+5s+s+5\right)}{(s+5)\cdot(s+1)\cdot(4s-3)}=$$

$$=\frac{A\cdot\left(4s^2+s-3\right)+B\cdot\left(4s^2+17s-15\right)+C\cdot\left(s^2+6s+5\right)}{(s+5)\cdot(s+1)\cdot(4s-3)}=$$

$$=\frac{s^2\cdot(4A+4B+C)+s\cdot(A+17B+6C)+(-3A-15B+5C)}{(s+5)\cdot(s+1)\cdot(4s-3)};$$

s^2 : $\quad 4A+4B+C=4,$

s^1 : $\quad A+17B+6C=25,$

s^0 : $\quad -3A-15B+5C=27.$

We have a system:

$$\begin{cases}4A+4B+C=4;\\ A+17B+6C=25;\\ -3A-15B+5C=27;\end{cases}\Rightarrow\begin{cases}C=4-4A-4B;\\ A+17B+6\cdot(4-4A-4B)=25;\\ -3A-15B+5C=27;\end{cases}\Rightarrow\begin{cases}C=4-4A-4B;\\ A+17B+24-24A-24B=25;\\ -3A-15B+5C=27;\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}C=4-4A-4B;\\ -23A-7B=1;\\ -3A-15B+5C=27;\end{cases}\Rightarrow\begin{cases}C=4-4A-4\cdot\left(-\frac{23}{7}A-\frac{1}{7}\right);\\ B=-\frac{23}{7}A-\frac{1}{7};\\ -3A-15B+5C=27;\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}C=\frac{64}{7}A+\frac{32}{7};\\ B=-\frac{23}{7}A-\frac{1}{7};\\ -3A-15\cdot\left(-\frac{23}{7}A-\frac{1}{7}\right)+5\cdot\left(\frac{64}{7}A+\frac{32}{7}\right)=27;\end{cases}\Rightarrow\begin{cases}C=\frac{64}{7}A+\frac{32}{7};\\ B=-\frac{23}{7}A-\frac{1}{7};\\ -3A+\frac{345}{7}A+\frac{15}{7}+\frac{320}{7}A+\frac{160}{7}=27;\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}C=\frac{64}{7}A+\frac{32}{7};\\ B=-\frac{23}{7}A-\frac{1}{7};\\ \frac{644}{7}A=\frac{14}{7};\end{cases}\Rightarrow\begin{cases}C=\frac{768}{161};\\ B=-\frac{3}{14};\\ A=\frac{1}{46}.\end{cases}$$

$$\frac{4s^2+25s+27}{(s+5)\cdot(s+1)\cdot(4s-3)}=\frac{A}{s+5}+\frac{B}{s+1}+\frac{C}{4s-3}=\frac{1}{46}\cdot\frac{1}{s+5}-\frac{3}{14}\cdot\frac{1}{s+1}+$$

$$+\frac{768}{161}\cdot\frac{1}{4s-3}=\frac{1}{46}\cdot\frac{1}{s+5}-\frac{3}{14}\cdot\frac{1}{s+1}+\frac{768}{161\cdot 4}\cdot\frac{1}{s-\frac{3}{4}}=\frac{1}{46}\cdot\frac{1}{s+5}-$$

$$-\frac{3}{14}\cdot\frac{1}{s+1}+\frac{192}{161}\cdot\frac{1}{s-\frac{3}{4}}.$$

$$Y(s)=\frac{1}{46}\cdot\frac{1}{s+5}-\frac{3}{14}\cdot\frac{1}{s+1}+\frac{192}{161}\cdot\frac{1}{s-\frac{3}{4}};$$

$$\frac{1}{s+5}\to e^{-5t},$$
$$\frac{1}{s+1}\to e^{-t},$$
$$\frac{1}{s-\frac{3}{4}}\to e^{\frac{3}{4}t}.$$
$$Y(s)\to y(t).$$

One finally gets

$$y(t)=\frac{1}{46}\cdot e^{-5t}-\frac{3}{14}\cdot e^{-t}+\frac{192}{161}\cdot e^{\frac{3}{4}t}.$$

Answer: $y(t)=\frac{1}{46}\cdot e^{-5t}-\frac{3}{14}\cdot e^{-t}+\frac{192}{161}\cdot e^{\frac{3}{4}t}.$