Answer on Question #82430 – Math – Differential Equations

Question

solve d^2u/dx^2 +d^2u/dy^2 = 0 which satisfies u (0,y) = u (l,y) = u (x, 0) =0 & u(x,a)= sin nπx/L

Solution

DE

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0;$$

Boundary conditions

$$u(0,y)=u(L,y)=u(x,0)=0;\ \ u(x,L)=\sin\frac{n\pi x}{L}.$$

This is Laplace's equation, which one can solve by the separation of variables.

Denote

$$u(x,y)=X(x)Y(y).$$

Then

$$\frac{\partial^2 (XY)}{\partial x^2}+\frac{\partial^2 (XY)}{\partial y^2}=0;$$

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0;$$

Divide by XY to get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0;$$

We can see that these two parts are functions of only x and only y. Because of (x,y) are independent variables, the first term must be a constant, and the second term must be a constant. We call them $-k^2$ and $+k^2$ (their sum must be null).

The new equations are

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-k^2;\qquad and \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k^2.$$

a) Solution for X:

$$\frac{\partial^2 X}{\partial x^2}=-k^2X;$$

$$X''+k^2X=0.$$

Solution is

$$X(x)=A\sin kx+B\cos kx.$$

Boundary conditions: $X(0) = 0;\ \ X(L) = 0.$

$$\begin{cases} A\sin k\cdot 0 + B\cos k\cdot 0 = 0; \\ A\sin kL + B\cos kl = 0; \end{cases} \Rightarrow \begin{cases} B = 0; \\ kL = n\pi, \qquad n \in Z. \end{cases}$$

$$u(x, L) = \sin\frac{n\pi x}{L} \ \Rightarrow \ A = 1.$$

$$X_n(x) = \sin\frac{n\pi}{L}x.$$

b) Solution for Y:

$$\frac{\partial^2 Y}{\partial y^2} = k^2 Y;$$

$$Y'' - k^2 Y = 0.$$

Solution is

$$Y(y) = Ce^{-ky} + De^{ky}.$$

Boundary conditions: $Y(0) = 0; Y(L) = 1.$

$$0 = C + D \ \Rightarrow \ C = -D;$$

$$Y(y) = C(e^{-ky} - e^{ky}) = C_1 \sinh ky\,; \quad where \quad C_1 = 2C.$$

$$1 = C_1 \sinh kL\,; \ \Rightarrow \ C_{1n} = \frac{1}{\sinh kL} = \frac{1}{\sinh n\pi}.$$

$$Y_n(y) = \frac{1}{\sinh n\pi}\sinh\frac{n\pi}{L}y.$$

So we can write the general solution. It must be combination of all solutions for $X(x)$ and $Y(y)$:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sinh n\pi}\sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right).$$