Answer on Question #82407 – Math – Abstract Algebra

Question

I Let G be a group, $H \trianglelefteq G$ and $\beta \leq G/H$. Let $A = \{x \in G \mid Hx \in \beta\}$. Show that

i) $A \leq G$

ii) $H \trianglelefteq A$

iii) $\beta = A/H$

Solution

Let f be homomorphism $G \to G/H$.

i) $A = \{x, f(x) \in \beta\}$ so $A = f^{-1}(\beta)$, so A is subgroup of G, because preimage of a subgroup is a subgroup.

ii) $H = f^{-1}(0)$, so $H \leq A$, and H is normal in G, so $\forall h \in H \text{ and } \forall g \in G \Rightarrow ghg^{-1} \in H$, so, even more so $\forall h \in H \text{ and } \forall a \in A \Rightarrow aha^{-1} \in H$, so H is normal in A: $H \trianglelefteq A$.

iii) Let $Ha \in A/H$, so $a \in A$, so $Ha \in \beta$ (by definition), so $A/H \subseteq \beta$.

And vice versa, $Hb \in \beta \implies b \in A \implies Hb \in A/H \implies \beta \subseteq A/H$

So $A/H \subseteq \beta$ and $\beta \subseteq A/H \implies \beta = A/H$.