Answer on the Question #82357 – Math – Combinatorics | Number Theory

Question

1. Let S= 1^1 + 2^2+3^3+.....+2010^2010. . What is the remainder when S is divided by 2?
2. Find the number of odd coefficients in expansion of (x + y)^2010.
3. Find all prime numbers p and integers a and b (not necessarily positive) such that p^a + p^b is the square of a rational number.

Solution

1) Odd number in power of the odd number is the odd number because it's just multiplication of several odd numbers which actually is odd.

Even number in power of the even number is the even number because it's just a product of several even numbers which actually is even.

When we divide S by 2 each even summand left remainder equals 0 and each odd summand left remainder 1.

From 1 to 2010 there is 2010 / 2 = 1005 odd numbers so sum of odd remainders equal 1005, so when we divide 1005 by 2 we get remainder 1 and this is the same as the remainder when S is divided by 2.

2) Firstly, note that the numerical coefficients of $(x+y)^{2010}$ are the same as those of $(x+1)^{2010}$, so I'm slightly simplifying your problem by instead considering the coefficients of x^k in the expansion of $(x+1)^{2010}$.
The coefficient of x^k in the expansion of $(x+1)^{2010}$ is **even** if and only if the binary representation of k has a 1 where the binary representation of the number 2010 has 0
This means that there are 1005+502+248=1755 coefficients in the expansion of $(x+1)^{2010}$ that are even. Thus, the number of coefficients in the expansion of (x+1)2010 that are odd is 2011−1755=256.
Here is full solution by using a theorem by Lucas. You can read it on this link
https://www.quora.com/How-many-odd-coefficients-are-there-in-the-expansion-of-x+y-2010

3) Take first a=b. Then we want $2p^a$ to be perfect square, which happens if p=2 and a is odd. That gives one infinite family of solutions. Now, without loss of generality we may take a<b. So we want $p^a(1+p^{b-a})$ to be a perfect square. Thus, a must be even and $1+p^{b-a}$ must be a perfect square. Let $p^{b-a}+1=x^2$. Then $p^{b-a}=(x-1)(x+1)$. If p is odd this forces p=3 and b−a=1. That gives the family of solutions p=3, a=2t, b=2t+1.If p=2, then x must be 3, for 3 is the only x such that x−1 and x+1 are powers of 2. That gives the family p=2, a=2t, b=2t+3.
Full answer is the third on link
https://math.stackexchange.com/questions/1012705/finding-all-prime-numbers-p-such-that-pa-pb-is-a-perfect-square

Answer: 1) 1; **2)** 256; **3)** p=3, a=2t, b=2t+1 or p=2, a=2t, b=2t+3.