Answer on Question #82353 – Math – Abstract Algebra

Question

Show that Z to Z/nZ is a natural epimorphism.

Solution

Not every homomorphism from Z to Z/nZ is a natural epimorphism.

Say, if n is composite, $n = a \cdot b$, and $a, b \neq 1, n$, let's consider $f: Z \to Z/nZ$ which maps every $z \in Z$ to the residue class $[a \cdot z] \in Z/nZ$.

It is not epimorphism, since

$$f(b) = [a \cdot b] = [n] = [0],$$

so $Im(f)$ is cyclic and its order equals b, whilst order of Z/nZ equals n, so they could not be isomorphic. So $Im(f) \neq Z/nZ$, so f is not epimorphism.

But if n is prime and $f: Z \to Z/nZ$ is not trivial, then f is always epimorphism, because $Im(f) \leq Z/nZ$, and since n is prime, so its only subgroups are $\{0\}$ and Z/nZ itself, and since f is not trivial, then $Im(f) = Z/nZ$, so f is epimorphism.