ANSWER on Question #82285 – Math – Differential Equations

QUESTION

Find a power series solution of the initial-value problem.

$$(x^2+1)y''(x)+xy'(x)+xy(x)=0,\ \ y(0)=3,\ \ y'(0)=-1$$

SOLUTION

Let

$$y(x)=\sum_{n=0}^{+\infty}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

$$y(0)=3=a_0+a_1\cdot 0+a_2\cdot 0^2+a_3\cdot 0^3+\cdots\to\boxed{a_0=3}$$

$$y'(0)=-1=a_1+2\cdot a_2\cdot 0+3\cdot a_3\cdot 0^2+\cdots\to\boxed{a_1=-1}$$

Then,

$$y(x)=\sum_{n=0}^{+\infty}a_nx^n\to y'(x)=\sum_{n=0}^{+\infty}a_n\cdot n\cdot x^{n-1}\equiv\sum_{n=1}^{+\infty}a_n\cdot n\cdot x^{n-1}=\begin{bmatrix}n-1=k\\ n=k+1\\ n=1\to k=0\end{bmatrix}\to$$

$$y'(x)=\sum_{k=0}^{+\infty}a_{(k+1)}\cdot(k+1)\cdot x^k\to\boxed{y'(x)=\sum_{n=0}^{+\infty}a_{(n+1)}\cdot(n+1)\cdot x^n}$$

$$y'(x)=\sum_{n=0}^{+\infty}a_{(n+1)}\cdot(n+1)\cdot x^n\to y''(x)=\sum_{n=0}^{+\infty}a_{(n+1)}\cdot(n+1)\cdot n\cdot x^{n-1}\equiv$$

$$\equiv\sum_{n=1}^{+\infty}a_{(n+1)}\cdot(n+1)\cdot n\cdot x^{n-1}=\begin{bmatrix}n-1=k\\ n=k+1\\ n=1\to k=0\end{bmatrix}=\sum_{k=0}^{+\infty}a_{(k+2)}\cdot(k+2)(k+1)\cdot x^k\to$$

$$y''(x)=\sum_{k=0}^{+\infty}a_{(k+2)}\cdot(k+2)(k+1)\cdot x^k\to\boxed{y''(x)=\sum_{n=0}^{+\infty}a_{(n+2)}\cdot(n+2)(n+1)\cdot x^n}$$

Substitute all the found series in the original equation.

$$(x^2+1)y''(x)+xy'(x)+xy(x)=0 \rightarrow$$

$$(x^2+1)\cdot\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^n + x\cdot\sum_{n=0}^{+\infty} a_{(n+1)}\cdot(n+1)\cdot x^n + x\cdot\sum_{n=0}^{+\infty} a_n x^n = 0 \rightarrow$$

$$\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^{n+2} + \sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^n +$$

$$+\sum_{n=0}^{+\infty} a_{(n+1)}\cdot(n+1)\cdot x^{n+1} + \sum_{n=0}^{+\infty} a_n x^{n+1} = 0$$

In these series are different degrees of x. Let us make sure that all ranks have the same degrees.

$$\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^{n+2} = \begin{bmatrix} n+2=k \\ n=k-2 \\ n=2\rightarrow k=2 \end{bmatrix} = \sum_{k=2}^{+\infty} a_k\cdot k(k-1)\cdot x^k \rightarrow$$

$$\boxed{\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^{n+2} = \sum_{k=2}^{+\infty} a_k\cdot k(k-1)\cdot x^k}$$

$$\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^n = a_{(0+2)}\cdot(0+2)(0+1)\cdot x^0 + a_{(1+2)}\cdot(1+2)(1+1)\cdot x^1 +$$

$$+\sum_{k=2}^{+\infty} a_{(k+2)}\cdot(k+2)(k+1)\cdot x^k = 2a_2+6a_3x+\sum_{k=2}^{+\infty} a_{(k+2)}\cdot(k+2)(k+1)\cdot x^k \rightarrow$$

$$\boxed{\sum_{n=0}^{+\infty} a_{(n+2)}\cdot(n+2)(n+1)\cdot x^n = 2a_2+6a_3x+\sum_{k=2}^{+\infty} a_{(k+2)}\cdot(k+2)(k+1)\cdot x^k}$$

$$\sum_{n=0}^{+\infty} a_{(n+1)}\cdot(n+1)\cdot x^{n+1} = \begin{bmatrix} n+1=m \\ n=m-1 \\ n=0\rightarrow m=1 \end{bmatrix} = \sum_{m=1}^{+\infty} a_m\cdot m\cdot x^m = a_1x+\sum_{m=2}^{+\infty} a_m\cdot m\cdot x^m \rightarrow$$

$$\boxed{\sum_{n=0}^{+\infty} a_{(n+1)}\cdot(n+1)\cdot x^{n+1} = a_1x+\sum_{k=2}^{+\infty} a_k\cdot k\cdot x^k}$$

$$\sum_{n=0}^{+\infty} a_n x^{n+1} = \begin{bmatrix} n+1=m \\ n=m-1 \\ n=0\rightarrow m=1 \end{bmatrix} = \sum_{m=1}^{+\infty} a_{(m-1)}x^m = a_{(1-1)}x+\sum_{m=2}^{+\infty} a_{(m-1)}x^m \rightarrow$$

$$\boxed{\sum_{n=0}^{+\infty} a_n x^{n+1} = a_0 x + \sum_{k=2}^{+\infty} a_{(k-1)} x^k}$$

Our equation takes the form

$$\sum_{k=2}^{+\infty} a_k \cdot k(k-1) \cdot x^k + 2a_2 + 6a_3 x + \sum_{k=2}^{+\infty} a_{(k+2)} \cdot (k+2)(k+1) \cdot x^k +$$

$$+a_1 x + \sum_{k=2}^{+\infty} a_k \cdot k \cdot x^k + a_0 x + \sum_{k=2}^{+\infty} a_{(k-1)} x^k = 0 \rightarrow$$

$$2a_2 + (6a_3 + a_1 + a_0)x + \sum_{k=2}^{+\infty} \left[k(k-1)a_k + ka_k + a_{(k-1)} + (k+2)(k+1)a_{(k+2)}\right] x^k = 0 \rightarrow$$

$$\begin{cases} 2a_2 = 0 \\ 6a_3 + a_1 + a_0 = 0 \\ k(k-1)a_k + ka_k + a_{(k-1)} + (k+2)(k+1)a_{(k+2)} = 0, \quad \forall k \geq 2 \end{cases} \rightarrow$$

$$\boxed{\begin{cases} a_0 = 3 \\ a_1 = -1 \\ a_2 = 0 \\ 6a_3 + 3 - 1 = 0 \rightarrow a_3 = -\dfrac{1}{3} \\ a_{(k+2)} = -\dfrac{k^2 a_k + a_{(k-1)}}{(k+1)(k+2)}, \quad \forall k \geq 2 \end{cases}}$$

Conclusion,

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \sum_{k=4}^{+\infty} a_k x^k = 3 - x - \frac{x^3}{3} + \sum_{k=4}^{+\infty} a_k x^k$$

$$\boxed{\begin{cases} a_0 = 3 \\ a_1 = -1 \\ a_2 = 0 \\ a_3 = -\dfrac{1}{3} \\ a_{(k+2)} = -\dfrac{k^2 a_k + a_{(k-1)}}{(k+1)(k+2)}, \quad \forall k \geq 2 \\ y(x) = 3 - x - \dfrac{x^3}{3} + \displaystyle\sum_{k=4}^{+\infty} a_k x^k \end{cases}}$$

ANSWER

$$\begin{cases} a_0 = 3 \\ a_1 = -1 \\ a_2 = 0 \\ a_3 = -\dfrac{1}{3} \\ a_{(k+2)} = -\dfrac{k^2 a_k + a_{(k-1)}}{(k+1)(k+2)}, \quad \forall k \geq 2 \\ y(x) = 3 - x - \dfrac{x^3}{3} + \displaystyle\sum_{k=4}^{+\infty} a_k x^k \end{cases}$$