Answer to the question 82223, Math / Functional Analysis

> Show that $p(x) = lim(x_n)$ where $x = (x_n)$ belongs to $l(\infty)$ defines a sublinear functional on $l(\infty)$

As I understood, $x = (x_1, x_2, \ldots) \in l_\infty$, $p(x) = \lim_{n\to\infty} x_n$. The functional $p(x)$ is defined on the set of all converging sequences and it is linear on the set. It is not defined for divergent sequences. So, strictly speaking the formula defines functional only on linear subspace of l_∞ of all converging sequences.

But the functional can be extended from linear subspace of l_∞ of all converging sequences to the entire l_∞.

As sublinear extension could be easily written out. We can take the functional $\overline{p}(x) = \limsup_{n\to\infty} x_n$. It is obvious that for converging sequence s the equality $\overline{p}(x) = p(x)$ holds. Also it is easy to see that for positive t holds $\limsup_{n\to\infty} tx_n = t \limsup_{n\to\infty} x_n$, $\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$. I. e. $\overline{p}$ is sublinear extension of p.

Moreover, by Hahn-Banach theorem there exists a linear extension.

A formulation of Hahn-Banach theorem: $p(x)$ is a sublinear functional defined on linear space V, U is linear subspace of V, $l(x)$ is a linear functional defined on U, $l(x) \le p(x)$ for all $x \in U$, then there exists a linear extension $\bar{l}$ of l, i.e. $\bar{l}(x) = l(x)$ for $x \in U$, $\bar{l}$ is linear, $\bar{l}$ is defined on V, $\bar{l}(x) \le p(x)$ for all $x \in V$.

As $p(x)$ we can take $\|x\|_{l_\infty}$ (every norm is sublinear functional as positive homogeneity and triangle inequality holds), i. e. by the definition of norm and the definition of sublinear functional. Also as $p(x)$ we can take $\overline{p}(x) = \limsup_{n\to\infty} x_n$. It is sublinear, and by definition the inequality $p(x) \le \overline{p}(x)$ holds.