Answer on Question #82164 – Math – Abstract Algebra

Question

Find unit set of $Z[\sqrt{m}]$, where m is not perfect square.

Solution

We have to find all integer solutions (x, y) to the Pell equation (1):

$a = x + y\sqrt{m};$

$b = x - y\sqrt{m};$

$ab = 1;$

$(x + y\sqrt{m})(x - y\sqrt{m}) = 1.$

$x^2 - my^2 = 1.$ (1)

1) $m = -1$:

$x^2 + y^2 = 1.$ (2)

This equation has four trivial solutions: $(\pm 1, 0)$, $(0, \pm 1)$ and 4 units:

$-1; 1; -i; i.$

2) $m < -1$, $m \in Z$:

$x^2 + (-m)y^2 = 1.$ (3)

In this case we have two trivial solutions: $(\pm 1, 0)$ and 2 units.

$-1; 1.$

3) In case of the positive non-perfect number for m we have two trivial solutions: $(\pm 1, 0)$, 2 units: -1; 1, and infinite number of non-trivial solutions:

$(x; y) = (\pm x_n; \pm y_n)$, where:

$x_n + y_n\sqrt{m} = (x_0 + y_0\sqrt{m})^n$, $n \in N$.

For example, if m = 2, 3, 5:

a) $m = 2$;

$x^2 - 2y^2 = 1$:

$x_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n$.

b) m = 3;

$x^2 - 3y^2 = 1$:

$x_n + y_n\sqrt{3} = (2 + \sqrt{3})^n$.

c) m = 5;

$x^2 - 5y^2 = 1$:

$x_n + y_n\sqrt{5} = (9 + 4\sqrt{5})^n$.