Answer on Question #81709 – Math– Combinatorics | Number Theory

Question

(x^1+x^2+............+x^100)^500 if you apply binomial theorem how many digits there will be

Solution

If by digit is meant any summand and coefficients are not put together (i.e. form $x^{500}+x^{500}+\ldots+x^{500}+x^{501}+\ldots+x^{501}+\ldots+x^{50000}+\ldots+x^{50000}$) then the solution is following.

There are 100 ways to choose the digit from the first multiplier $x+x^2+\ldots+x^{100}$, 100 ways to choose the digit from the second multiplier, and so on up to 100 ways to choose the digit from the 500th multiplier. Then the total number of ways is

$\underbrace{100\cdot 100\cdot\ldots\cdot 100}_{500\ \ times}=100^{500}$, it is the number of digits.

I thought that the question is about the number of digits which arise as powers (because the binomial theorem gives a form of sum with coefficients).

Then it was my solution:

The minimum power of x in the expansion of $\left(x+x^2+x^3+\ldots+x^{100}\right)^{500}$ is 500: $x\cdot x\cdot x\cdot\ldots\cdot x=x^{500}$, the maximum power of x in the expansion is 50000: $x^{100}\cdot x^{100}\cdot\ldots\cdot x^{100}=\left(x^{100}\right)^{500}=x^{50000}$.

Let us prove that for any $n\in\mathbb{Z}$: $500\le n<50000$ x^n is present in the expansion. We have

$$\left(x+x^2+x^3+\ldots+x^{100}\right)^{500}=x^{500}\left(1+x+x^2+\ldots+x^{99}\right)^{500}$$

Then we have to prove that for any $0\le m<49500$ x^m is present in the expansion of $\left(1+x+x^2+\ldots+x^{99}\right)^{500}$.

Divide m by 99:

$m=99k+l$

where
$0\le k<500, 0\le l<99$.

Then x^m can be represented as $x^m=\underbrace{x^{99}\cdot x^{99}\cdot\ldots\cdot x^{99}}_{k\ \ times}\cdot x^l\cdot\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{500-1-k\ge 0\ \ times}$, thus it is present in the expansion.

Then the number of digits is 50000-500+1=49501 (we add 1 since we need to include both numbers 500 and 50000, and 50000-500 is number of numbers between 500 and 50000 not including 50000).