Answer on Question #81521- Math – Differential Equations

Solve the following ODE using the power series method:
(x + 2)y'' + xy' − y = 0

Solution

We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$y' = \sum_{n=1}^{\infty} n \cdot c_n x^{n-1},$$

and

$$y'' = \sum_{n=2}^{\infty} (n-1)n \cdot c_n x^{n-2}$$

Substituting in the differential equation, we get

$$(x + 2)\sum_{n=2}^{\infty}(n-1)n \cdot c_n x^{n-2} + x\sum_{n=1}^{\infty} n \cdot c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$x\sum_{n=2}^{\infty}(n-1)n \cdot c_n x^{n-2} + 2\sum_{n=2}^{\infty}(n-1)n \cdot c_n x^{n-2} + \sum_{n=1}^{\infty} n \cdot c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty}(n-1)n \cdot c_n x^{n-1} + 2\sum_{n=2}^{\infty}(n-1)n \cdot c_n x^{n-2} + \sum_{n=1}^{\infty} n \cdot c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty}(n+1)n \cdot c_{n+1} x^n + 2\sum_{n=0}^{\infty}(n+2)(n+1) \cdot c_{n+2} x^n + \sum_{n=0}^{\infty}(n-1) \cdot c_n x^n = 0$$

$$\sum_{n=1}^{\infty}[(n+1)n \cdot c_{n+1} + 2(n+1)(n+2) \cdot c_{n+2} + (n-1) \cdot c_n]x^n = 0$$

This equation is true if the coefficient of x^n is 0:

$$\sum_{n=1}^{\infty}[(n+1)n \cdot c_{n+1} + 2(n+1)(n+2) \cdot c_{n+2} + (n-1) \cdot c_n]x^n = 0$$

$$n(n+1) \cdot c_{n+1} + 2(n+1)(n+2) \cdot c_{n+2} + (n-1) \cdot c_n = 0$$

$$c_{n+2}=\frac{(1-n)\cdot c_n-n(n+1)\cdot c_{n+1}}{2(n+1)(n+2)},\qquad n=0,1,2,3\ldots$$

Put $n=0,1,2,3\ldots$

$n=0$:

$$c_2=\frac{c_0}{2\cdot 1\cdot 2}=\frac{c_0}{4}$$

$n=1$:

$$c_3=\frac{-1\cdot 2\cdot c_2}{2\cdot 2\cdot 3}=-\frac{c_2}{2\cdot 3}=-\frac{c_0}{2\cdot 3\cdot 4}=-\frac{c_0}{4!}$$

$n=2$:

$$c_4=\frac{-1\cdot c_2-2\cdot 3c_3}{2\cdot 3\cdot 4}=\frac{-\frac{c_0}{4}-6\cdot\left(-\frac{c_0}{2\cdot 3\cdot 4}\right)}{2\cdot 3\cdot 4}=0$$

$n=3$:

$$c_5=\frac{-2\cdot c_3-3\cdot 4\,c_4}{2\cdot 4\cdot 5}=\frac{-2\cdot\left(-\frac{c_0}{4!}\right)}{2\cdot 4\cdot 5}=\frac{c_0}{4\cdot 5!}$$

$n=4$:

$$c_6=\frac{-3\cdot c_4-4\cdot 5\,c_5}{2\cdot 5\cdot 6}=\frac{-4\cdot 5\,\frac{c_0}{4\cdot 5!}}{2\cdot 5\cdot 6}=-\frac{c_0}{2\cdot 6!}$$

The solution is

$$y=c_0+c_1x+c_2x^2+c_3x^3+\cdots=$$

$$=c_0\left(1+\frac{1}{4}x^2-\frac{1}{4!}x^3+\frac{1}{4\cdot 5!}x^5-\frac{1}{2\cdot 6!}x^6+\cdots\right)$$