ANSWER on Question #81519 – Math – Differential Equations

QUESTION

Reduce the following PDE to a set of three ODEs by the method of separation of variables

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial V}{\partial r}\right)\right)+\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial^2 V}{\partial z^2}+k^2\cdot V=0$$

SOLUTION

Assume

$$V(r,\theta,z)=R(r)\Theta(\theta)Z(z)$$

Then,

$$\Theta Z\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial R}{\partial r}\right)\right)+RZ\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}+R\Theta\frac{\partial^2 Z}{\partial z^2}+k^2\cdot R\Theta Z=0\Bigg|\div\left(\frac{1}{R\Theta Z}\right)\rightarrow$$

$$\frac{1}{\mathrm{R}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial R}{\partial r}\right)\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k^2=0\rightarrow$$

$$\frac{1}{\mathrm{R}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial R}{\partial r}\right)\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}+k^2=-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

In the above equation the left-hand side depends on r and θ, while the right-hand side depends on z. The only way these two members are going to be equal for all values of r, θ and z is when both of them are equal to a constant. Let us define such constant as $-l^2$.

With this choice for the constant, we obtain:

$$\frac{d^2 Z}{dz^2}-l^2\cdot Z=0$$

The general solution of this equation is:

$$Z(z)=A_1e^{lz}+A_2e^{-lz}$$

Such a solution, when considering the specific boundary conditions, will allow $Z(z)$ to go to zero for z going to $\pm\infty$, which makes physical sense. If we had given the constant a value of l^2, we would have had periodic trigonometric functions, which do not tend to zero for z going to infinity.

Once sorted the z −dependency, we need take care of r and θ.

$$\frac{1}{\mathrm{R}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial R}{\partial r}\right)\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}+k^2=-l^2\rightarrow$$

$$\frac{1}{\mathrm{R}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\left(\frac{\partial R}{\partial r}\right)\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}=-(k^2+l^2)\rightarrow$$

$$\frac{1}{R}\frac{1}{r}\left(\frac{\partial R}{\partial r}+r\cdot\frac{\partial^2 R}{\partial r^2}\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}=-(k^2+l^2)\rightarrow$$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{R}\frac{1}{r}\frac{\partial R}{\partial r}+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\cdot\frac{\partial^2\Theta}{\partial\theta^2}=-(k^2+l^2)\Big|\times(r^2)\rightarrow$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+(k^2+l^2)r^2=\frac{1}{\Theta}\cdot\frac{\partial^2\Theta}{\partial\theta^2}$$

Again we are in a situation where the only way a solution can be found for the above equation is when both members are equal to a constant. This time we select a positive constant, which we call m^2. The equation for Θ becomes, for:

$$\frac{d^2\Theta}{d\theta^2}+m^2\Theta=0$$

This solution is well suited, to describe the variation for an angular coordinate like θ. Had we chosen the set both members equal to a negative number, we would have ended up with exponential functions with a different value assigned to $\Theta(\theta)$ for each $360°$ turn, a clear non-physical solution.

Last to be examined is the r −dependency. We have:

$$\frac{r^2}{R}\frac{d^2 R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+(k^2+l^2)r^2=m^2\rightarrow$$

$$r^2\cdot\frac{d^2R}{dr^2}+r\cdot\frac{dR}{dr}+[(k^2+l^2)r^2-m^2]\cdot R=0$$

This equation is a well-known equation of mathematical physics called parametric Bessel's equation. With sample linear transformation of variable, $x = r \cdot \sqrt{(k^2 + l^2)}$, equation is readily changed into a Bessel's equation:

$$\frac{dR}{dr} = \frac{dR}{dx} \cdot \frac{dx}{dr} = \sqrt{k^2 + l^2} \cdot R'$$

$$\frac{d^2R}{dr^2} = \frac{d}{dx}\left(\sqrt{k^2 + l^2} \cdot R'\right) \cdot \frac{dx}{dr} = (k^2 + l^2) \cdot R''$$

Then,

$$r^2 \cdot \frac{d^2R}{dr^2} + r \cdot \frac{dR}{dr} + [(k^2 + l^2)r^2 - m^2] \cdot R = 0 \rightarrow$$

$$\frac{x^2}{(k^2 + l^2)} \cdot (k^2 + l^2) \cdot R'' + \frac{x}{\sqrt{k^2 + l^2}} \cdot \sqrt{k^2 + l^2} \cdot R' + [x^2 - m^2] \cdot R = 0 \rightarrow$$

$$x^2 \cdot R'' + x \cdot R' + [x^2 - m^2] \cdot R = 0$$

where R'' and R' indicate the first and second derivatives with respect to x.

In what follows we will assume that m is a real, non-negative number.

Linearly independent solutions are typically denote $J_m(x)$ (Bessel Functions) and $N_m(x)$(Neumann Functions).