Answer on Question #81486 – Math – Linear Algebra

Question

Find the orthogonal canonical reduction of the quadratic form

$$x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

Also, find its principal axes.

Solution

$$q = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

Matrix of quadratic form

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$q = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix}$$

Characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 1-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & 1-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$(1-\lambda)(1 - 2\lambda + \lambda^2 - 1) + (-1 + \lambda - 1) - (1 + 1 - \lambda) = 0$$

$$-2\lambda + \lambda^2 + 2\lambda^2 - \lambda^3 - 2 + \lambda - 2 + \lambda = 0$$

$$-\lambda^3 + 3\lambda^2 - 4 = 0$$

$$-(1+\lambda)\lambda^2 + 4(\lambda^2 - 1) = 0$$

$$-(1+\lambda)(\lambda^2 - 4\lambda + 4) = 0$$

$$-(1+\lambda)(\lambda - 2)^2 = 0$$

$\lambda_1 = 2$
$\lambda_2 = 2$
$\lambda_3 = -1$
These are eigenvalues.
Find the eigenvectors
$\lambda = 2$

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1)R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $v_2 = t, v_3 = s$ then $v_1 = -t - s$

Therefore,

$$\mathbf{v} = \begin{pmatrix} -t-s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} s$$

$\lambda = -1$

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 + \left(\frac{1}{2}\right)R_1} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_3 + \left(\frac{1}{2}\right)R_1} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\left(\frac{2}{3}\right)R_2} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $v_3 = t$ then $v_1 = t, v_2 = t$

Therefore,

$$\mathbf{v} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t$$

Eigenvalue:2, eigenvectors: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Eigenvalue:-1, eigenvector: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Thus, the orthogonal canonical reduction is

$$q = \begin{pmatrix} x' & y' & z' \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 2(x')^2 + 2(y')^2 - (z')^2$$

The normed vector are principal axes

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{and } \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$