Answer on Question #81451 – Math – Linear Algebra
Question

Check whether the matrices A and B are diagonalisable. Diagonalise those matrices which are diagonalisable.

i) $A = \begin{pmatrix} -2 & -5 & -1 \\ 3 & 6 & 1 \\ -2 & -3 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} -2-\lambda & -5 & -1 \\ 3 & 6-\lambda & 1 \\ -2 & -3 & 1-\lambda \end{pmatrix}$$

Characteristic equation

$$\begin{vmatrix} -2-\lambda & -5 & -1 \\ 3 & 6-\lambda & 1 \\ -2 & -3 & 1-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\begin{vmatrix} 6-\lambda & 1 \\ -3 & 1-\lambda \end{vmatrix} - (-5)\begin{vmatrix} 3 & 1 \\ -2 & 1-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 3 & 6-\lambda \\ -2 & -3 \end{vmatrix} = 0$$

$$(-2-\lambda)(6-6\lambda-\lambda+\lambda^2+3)+5(3-3\lambda+2)-(-9+12-2\lambda)=0$$

$$-18+14\lambda-2\lambda^2-9\lambda+7\lambda^2-\lambda^3+25-15\lambda-3+2\lambda=0$$

$$-\lambda^3+5\lambda^2-8\lambda+4=0$$
$$(1-\lambda)\lambda^2+4(\lambda^2-2\lambda+1)=0$$
$$(1-\lambda)\lambda^2+4(1-\lambda)^2=0$$
$$(1-\lambda)(\lambda^2-4\lambda+4)=0$$
$$(1-\lambda)(\lambda-2)^2=0$$

The eigenvalues are
$\lambda_1 = 1,$
$\lambda_2 = 2,$
$\lambda_3 = 2.$
Since the eigenvalue 2 is repeated (it is algebraically degenerate) there is a possibility that there may not be enough independent eigenvectors to form a diagonalizing matrix.
Find the eigenvectors
$\lambda = 1$

$$\begin{pmatrix} -2-\lambda & -5 & -1 \\ 3 & 6-\lambda & 1 \\ -2 & -3 & 1-\lambda \end{pmatrix} = \begin{pmatrix} -3 & -5 & -1 \\ 3 & 5 & 1 \\ -2 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -5 & -1 \\ 3 & 5 & 1 \\ -2 & -3 & 0 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} -3 & -5 & -1 \\ 0 & 0 & 0 \\ -2 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix}-3 & -5 & -1\\ 0 & 0 & 0\\ -2 & -3 & 0\end{pmatrix} \xrightarrow{R_3-\left(\frac{2}{3}\right)R_1} \begin{pmatrix}-3 & -5 & -1\\ 0 & 0 & 0\\ 0 & 1/3 & 2/3\end{pmatrix}$$

Swap rows 2 and 3

$$\begin{pmatrix}-3 & -5 & -1\\ 0 & 1/3 & 2/3\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -5 & -1\\ 0 & 1/3 & 2/3\\ 0 & 0 & 0\end{pmatrix} \xrightarrow{(3)R_2} \begin{pmatrix}-3 & -5 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -5 & -1\\ 0 & 1/3 & 2/3\\ 0 & 0 & 0\end{pmatrix} \xrightarrow{R_1+(5)R_2} \begin{pmatrix}-3 & 0 & 9\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & 0 & 9\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} \xrightarrow{\left(-\frac{1}{3}\right)R_1} \begin{pmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

If we take $v_3 = t$, then $v_1 = 3t, v_2 = -2t$

Therefore,

$$\mathbf{v} = \begin{pmatrix}3t\\ -2t\\ t\end{pmatrix} = \begin{pmatrix}3\\ -2\\ 1\end{pmatrix}t$$

$\lambda = 2$

$$\begin{pmatrix}-2-\lambda & -5 & -1\\ 3 & 6-\lambda & 1\\ -2 & -3 & 1-\lambda\end{pmatrix} = \begin{pmatrix}-4 & -5 & -1\\ 3 & 4 & 1\\ -2 & -3 & -1\end{pmatrix}$$

$$\begin{pmatrix}-4 & -5 & -1\\ 3 & 4 & 1\\ -2 & -3 & -1\end{pmatrix} \xrightarrow{R_2+\left(\frac{3}{4}\right)R_1} \begin{pmatrix}-4 & -5 & -1\\ 0 & 1/4 & 1/4\\ -2 & -3 & -1\end{pmatrix}$$

$$\begin{pmatrix}-4 & -5 & -1\\ 0 & 1/4 & 1/4\\ -2 & -3 & -1\end{pmatrix} \xrightarrow{R_3-\left(\frac{1}{2}\right)R_1} \begin{pmatrix}-4 & -5 & -1\\ 0 & 1/4 & 1/4\\ 0 & -1/2 & -1/2\end{pmatrix}$$

$$\begin{pmatrix}-4 & -5 & -1\\ 0 & 1/4 & 1/4\\ 0 & -1/2 & -1/2\end{pmatrix} \xrightarrow{R_3+(2)R_1} \begin{pmatrix}-4 & -5 & -1\\ 0 & 1/4 & 1/4\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix} -4 & -5 & -1 \\ 0 & 1/4 & 1/4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(4)R_2} \begin{pmatrix} -4 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+(5)R_2} \begin{pmatrix} -4 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\left(-\frac{1}{4}\right)R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $v_3 = t$, then $v_1 = t, v_2 = -t$
Therefore,

$$\mathbf{v} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t$$

Eigenvalue:1, eigenvector: $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Eigenvalue:2, eigenvector: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Since the number of eigenvectors is less than dimension of the matrix, then the matrix is not diagonalisable.
The matrix A is not diagonalisable.

Question

ii) $B = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

Solution

$$\begin{pmatrix} -1-\lambda & -3 & 0 \\ 2 & 4-\lambda & 0 \\ -1 & -1 & 2-\lambda \end{pmatrix}$$

Characteristic equation

$$\begin{vmatrix} -1-\lambda & -3 & 0 \\ 2 & 4-\lambda & 0 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)\begin{vmatrix}4-\lambda & 0\\ -1 & 2-\lambda\end{vmatrix}-(-3)\begin{vmatrix}2 & 0\\ -1 & 2-\lambda\end{vmatrix}+0=0$$

$$(-1-\lambda)(8-4\lambda-2\lambda+\lambda^2+0)+3(4-2\lambda+0)+0=0$$

$$-8+6\lambda-\lambda^2-8\lambda+6\lambda^2-\lambda^3+12-6\lambda=0$$

$$-\lambda^3+5\lambda^2-8\lambda+4=0$$
$$(1-\lambda)\lambda^2+4(\lambda^2-2\lambda+1)=0$$
$$(1-\lambda)\lambda^2+4(1-\lambda)^2=0$$
$$(1-\lambda)(\lambda^2-4\lambda+4)=0$$
$$(1-\lambda)(\lambda-2)^2=0$$

The eigenvalues are
$\lambda_1=1$
$\lambda_2=2$
$\lambda_3=2$
Find the eigenvectors
$\lambda=1$

$$\begin{pmatrix}-1-\lambda & -3 & 0\\ 2 & 4-\lambda & 0\\ -1 & -1 & 2-\lambda\end{pmatrix}=\begin{pmatrix}-2 & -3 & 0\\ 2 & 3 & 0\\ -1 & -1 & 1\end{pmatrix}$$

$$\begin{pmatrix}-2 & -3 & 0\\ 2 & 3 & 0\\ -1 & -1 & 1\end{pmatrix}\xrightarrow{R_2+R_1}\begin{pmatrix}-2 & -3 & 0\\ 0 & 0 & 0\\ -1 & -1 & 1\end{pmatrix}$$

$$\begin{pmatrix}-2 & -3 & 0\\ 0 & 0 & 0\\ -1 & -1 & 1\end{pmatrix}\xrightarrow{R_3-\left(\frac{1}{2}\right)R_1}\begin{pmatrix}-2 & -3 & 0\\ 0 & 0 & 0\\ 0 & 1/2 & 1\end{pmatrix}$$

Swap rows 2 and 3

$$\begin{pmatrix}-2 & -3 & 0\\ 0 & 1/2 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-2 & -3 & 0\\ 0 & 1/2 & 1\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{(2)R_2}\begin{pmatrix}-2 & -3 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-2 & -3 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{R_1+(3)R_2}\begin{pmatrix}-2 & 0 & 6\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-2 & 0 & 6\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{\left(-\frac{1}{2}\right)R_1}\begin{pmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

If we take $v_3 = t$, then $v_1 = 3t, v_2 = -2t$
Therefore,

$$\mathbf{v}=\begin{pmatrix}3t\\ -2t\\ t\end{pmatrix}=\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}t$$

$\lambda = 2$

$$\begin{pmatrix}-1-\lambda & -3 & 0\\ 2 & 4-\lambda & 0\\ -1 & -1 & 2-\lambda\end{pmatrix}=\begin{pmatrix}-3 & -3 & 0\\ 2 & 2 & 0\\ -1 & -1 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -3 & 0\\ 2 & 2 & 0\\ -1 & -1 & 0\end{pmatrix}\xrightarrow{R_2+\left(\frac{2}{3}\right)R_1}\begin{pmatrix}-3 & -3 & 0\\ 0 & 0 & 0\\ -1 & -1 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -3 & 0\\ 0 & 0 & 0\\ -1 & -1 & 0\end{pmatrix}\xrightarrow{R_3-\left(\frac{1}{3}\right)R_1}\begin{pmatrix}-3 & -3 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -3 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{(-3)R_1}\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Solve the matrix equation

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

If we take $v_2 = t, v_3 = s$ then $v_1 = -t$
Therefore,

$$\mathbf{v}=\begin{pmatrix}-t\\ t\\ s\end{pmatrix}=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}t+\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}s$$

Eigenvalue:1, eigenvector: $\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}$

Eigenvalue:2, eigenvectors: $\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

Form the matrix P

$$P=\begin{pmatrix}3 & -1 & 0\\ -2 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

Form the diagonal matrix D

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$B = PDP^{-1}$

Question

b) Find inverse of the matrix B in part a) of the question by finding the adjoint as well as using Cayley-Hamiltion theorem.

Solution

$$B = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\det(B) = \begin{vmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} = 2(-4+6) = 4 \neq 0$$

$$C_{11} = \begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix} = 8, \quad C_{12} = -\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = -4, \quad C_{13} = \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} -3 & 0 \\ -1 & 2 \end{vmatrix} = 6, \quad C_{22} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = -2, \quad C_{23} = -\begin{vmatrix} -1 & -3 \\ -1 & -1 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -3 & 0 \\ 4 & 0 \end{vmatrix} = 0, \quad C_{32} = -\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} = 0, \quad C_{33} = \begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} = 2$$

$$Agj(B) = C^T = \begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det(B)} Agj(B) = \frac{1}{4}\begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

Augment the matrix with identity matrix

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(2)R_1} \begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{(-1)R_1} \begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2/(-2)} \begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_1-(3)R_2} \begin{pmatrix} 1 & 0 & 0 & 2 & 3/2 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 3/2 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_3/2} \begin{pmatrix} 1 & 0 & 0 & 2 & 3/2 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

We have obtained the identity matrix to the left. So, we are done.

$$B^{-1} = \begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

Use Cayley-Hamiltion theorem

$$p(t) = \det(B - tI)$$

$$\det(B - tI) = \begin{vmatrix} -1-t & -3 & 0 \\ 2 & 4-t & 0 \\ -1 & -1 & 2-t \end{vmatrix} = (2-t)\begin{vmatrix} -1-t & -3 \\ 2 & 4-t \end{vmatrix} =$$

$= (2-t)(-4 + t - 4t + t^2 + 6) =$

$= 4 - 6t + 2t^2 - 2t + 3t^2 - t^3 = -t^3 + 5t^2 - 8t + 4$

$p(B) = 0 => -B^3 + 5B^2 - 8B + 4I = 0$

$I = \frac{1}{4}(B^3 - 5B^2 + 8B)$

$I = \frac{1}{4}B(B^2 - 5B + 8I)$

$B^{-1} = \frac{1}{4}(B^2 - 5B + 8I)$

$$B^2 = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix} =$$

$$=\begin{pmatrix} -1(-1)-3(2)+0 & -1(-3)-3(4)+0 & 0 \\ 2(-1)+4(2)+0 & 2(-3)+4(4)+0 & 0 \\ -1(-1)-1(2)+2(-1) & -1(-3)-1(4)+2(-1) & 0+0+2(2) \end{pmatrix}=$$

$$=\begin{pmatrix} -5 & -9 & 0 \\ 6 & 10 & 0 \\ -3 & -3 & 4 \end{pmatrix}$$

$$B^2-5B+8I=\begin{pmatrix} -5 & -9 & 0 \\ 6 & 10 & 0 \\ -3 & -3 & 4 \end{pmatrix}+\begin{pmatrix} 5 & 15 & 0 \\ -10 & -20 & 0 \\ 5 & 5 & -10 \end{pmatrix}+\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}=$$

$$=\begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

$$B^{-1}=\frac{1}{4}\begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix}=\begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$B^{-1}=\begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}.$$