Answer on Question #81450 – Math – Linear Algebra

Question

ii) $B = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

b) Find inverse of the matrix B in part a) of the question by finding the adjoint as well as using Cayley-Hamiltion theorem.

Solution

$$B = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\det(B) = \begin{vmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} = 2(-4+6) = 4 \neq 0$$

$$C_{11} = \begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix} = 8, \quad C_{12} = -\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = -4, \quad C_{13} = \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} -3 & 0 \\ -1 & 2 \end{vmatrix} = 6, \quad C_{22} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = -2, \quad C_{23} = -\begin{vmatrix} -1 & -3 \\ -1 & -1 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -3 & 0 \\ 4 & 0 \end{vmatrix} = 0, \quad C_{32} = -\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} = 0, \quad C_{33} = \begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} = 2$$

$$Agj(B) = C^T = \begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det(B)} Agj(B) = \frac{1}{4}\begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

Use Cayley-Hamiltion theorem

$$p(t) = \det(B - tI)$$

$$\det(B - tI) = \begin{vmatrix} -1-t & -3 & 0 \\ 2 & 4-t & 0 \\ -1 & -1 & 2-t \end{vmatrix} = (2-t)\begin{vmatrix} -1-t & -3 \\ 2 & 4-t \end{vmatrix} =$$

$= (2-t)(-4+t-4t+t^2+6) =$

$= 4 - 6t + 2t^2 - 2t + 3t^2 - t^3 = -t^3 + 5t^2 - 8t + 4$

$p(B) = 0 => -B^3 + 5B^2 - 8B + 4I = 0$

$I = \frac{1}{4}(B^3 - 5B^2 + 8B)$

$I = \frac{1}{4}B(B^2 - 5B + 8I)$

$$B^{-1} = \frac{1}{4}(B^2 - 5B + 8I)$$

$$B^2 = \begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -3 & 0 \\ 2 & 4 & 0 \\ -1 & -1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -1(-1) - 3(2) + 0 & -1(-3) - 3(4) + 0 & 0 \\ 2(-1) + 4(2) + 0 & 2(-3) + 4(4) + 0 & 0 \\ -1(-1) - 1(2) + 2(-1) & -1(-3) - 1(4) + 2(-1) & 0 + 0 + 2(2) \end{pmatrix} =$$

$$= \begin{pmatrix} -5 & -9 & 0 \\ 6 & 10 & 0 \\ -3 & -3 & 4 \end{pmatrix}$$

$$B^2 - 5B + 8I = \begin{pmatrix} -5 & -9 & 0 \\ 6 & 10 & 0 \\ -3 & -3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 15 & 0 \\ -10 & -20 & 0 \\ 5 & 5 & -10 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} =$$

$$= \begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{4}\begin{pmatrix} 8 & 6 & 0 \\ -4 & -2 & 0 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 2 & 3/2 & 0 \\ -1 & -1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

Augment the matrix with identity matrix

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(2)R_1} \begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{(-1)R_1} \begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1 & 3 & 0 & -1 & 0 & 0\\ 0 & -2 & 0 & 2 & 1 & 0\\ 0 & 2 & 2 & -1 & 0 & 1\end{pmatrix}\xrightarrow{R_3+R_2}\begin{pmatrix}1 & 3 & 0 & -1 & 0 & 0\\ 0 & -2 & 0 & 2 & 1 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 3 & 0 & -1 & 0 & 0\\ 0 & -2 & 0 & 2 & 1 & 0\\ 0 & 2 & 2 & -1 & 0 & 1\end{pmatrix}\xrightarrow{R_2/(-2)}\begin{pmatrix}1 & 3 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & -1 & -1/2 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 3 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & -1 & -1/2 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{pmatrix}\xrightarrow{R_1-(3)R_2}\begin{pmatrix}1 & 0 & 0 & 2 & 3/2 & 0\\ 0 & 1 & 0 & -1 & -1/2 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 2 & 3/2 & 0\\ 0 & 1 & 0 & -1 & -1/2 & 0\\ 0 & 0 & 2 & 1 & 1 & 1\end{pmatrix}\xrightarrow{R_3/2}\begin{pmatrix}1 & 0 & 0 & 2 & 3/2 & 0\\ 0 & 1 & 0 & -1 & -1/2 & 0\\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2\end{pmatrix}$$

We have obtained the identity matrix to the left. So, we are done.

$$B^{-1}=\begin{pmatrix}2 & 3/2 & 0\\ -1 & -1/2 & 0\\ 1/2 & 1/2 & 1/2\end{pmatrix}$$