Answer on Question #81428, Math/Real Analysis

Show that the functions $f(x,y) = \ln x - \ln y$ and $g(x,y) = (x^2 + 2y^2)/2xy$ are functionally dependent.

The answer: By definition, two functions $f(x,y)$ and $g(x,y)$ defined in a domain D are functionally dependent of there exists a continuously differentiable function $\phi(u,v)$ such that :

- $\phi(f(x,y), g(x,y)) \equiv 0$
- $\Delta\phi \neq 0$ at each point $(u,v) = (f(x,y), g(x,y))$ for (x,y) in D.

  Then following the **theorem**: *'If F is differentiable transformation defined as*

$$F : \begin{cases} u = f(x,y) \\ v = g(x,y) \end{cases}$$

and f and g are linearly dependent, then the Jacobian of F is identically zero'

So in our case $f(x,y) = \ln x - \ln y$ and $g(x,y) = (x^2 + 2y^2)/2xy$, i.e. $x, y > 0$. To get the Jacobian one needs to calculate the partial derivatives: f_x, f_y, g_x, g_y:

$$\begin{aligned} f_x &= \frac{1}{x} ., \quad f_y = -\frac{1}{y} \\ g_x &= \frac{2x * 2xy - 2y * (x^2 + 2y^2)}{4x^2y^2} = \frac{x^2 - 2y^2}{2x^2y} \\ g_y &= \frac{4y * 2xy - 2x(x^2 + 2y^2)}{4x^2y^2} = \frac{-x^2 + 2y^2}{2xy^2} \end{aligned}$$

then

$$\det J = det\left[\frac{d(f,g)}{d(x,y)}\right] = f_x g_y - f_y g_x = \frac{-x^2 + 2y^2}{2x^2y^2} + \frac{x^2 - 2y^2}{2x^2y^2} = 0$$

Therefore the functions $f(x,y) = \ln x - \ln y$ and $g(x,y) = (x^2 + 2y^2)/2xy$ are functionally dependent.

As $f(x,y) = \ln x - \ln y = \ln \frac{x}{y}$ then $\frac{x}{y} = e^{f(x,y)}$. Therfore

$$g(x,y) = \frac{(x^2 + 2y^2)}{2xy} = \frac{1}{2}\frac{x}{y} + \frac{y}{x} = \frac{1}{2}e^{f(x,y)} + \frac{1}{e^{f(x,y)}}$$

that confirms a functional dependence of f and g functions.