

Answer on Question #81367 – Math – Linear Algebra

Question

Define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x, y, z) = (x + y, y, 2x - 2y + 2z)$. Check that T satisfies the polynomial $(x - 1)^2(x - 2)$. Find the minimal polynomial of T .

Solution

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix}$$

Find $(T - I)^2(T - 2I)$

$$T - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned} (T - I)^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 0(0) + 1(0) + 0(2) & 0(1) + 1(0) + 0(-2) & 0(0) + 1(0) + 0(1) \\ 0(0) + 0(0) + 0(2) & 0(1) + 0(0) + 0(-2) & 0(0) + 0(0) + 0(1) \\ 2(0) - 2(0) + 1(2) & 2(1) - 2(0) + 1(-2) & 2(0) - 2(0) + 1(1) \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$T - 2I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 2 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{aligned} (T - I)^2(T - 2I) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0(-1) + 0(0) + 0(2) & 0(1) + 0(-1) + 0(-2) & 0(0) + 0(0) + 0(0) \\ 0(-1) + 0(0) + 0(2) & 0(1) + 0(-1) + 0(-2) & 0(0) + 0(0) + 0(0) \\ 2(-1) + 0(0) + 1(2) & 2(1) + 0(-1) + 1(-2) & 2(0) + 0(0) + 1(0) \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Thus, the characteristic polynomial is obtained as

$$p(\lambda) = \det(T - \lambda I) = (\lambda - 1)^2(\lambda - 2)$$

$$\begin{aligned}
(T - I)(T - 2I) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} = \\
&= \begin{bmatrix} 0(-1) + 1(0) + 0(2) & 0(1) + 1(-1) + 0(-2) & 0(0) + 1(0) + 0(0) \\ 0(-1) + 0(0) + 0(2) & 0(1) + 0(-1) + 0(-2) & 0(0) + 0(0) + 0(0) \\ 2(-1) - 2(0) + 1(2) & 2(1) - 2(-1) + 1(-2) & 2(0) - 2(0) + 1(0) \end{bmatrix} = \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

The minimal polynomial of T

$$m_T(x) = (x - 1)^2(x - 2)$$